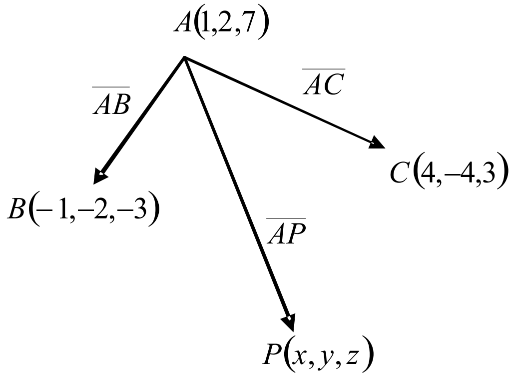
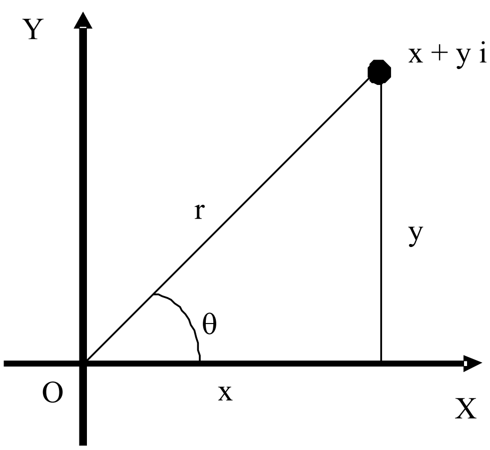
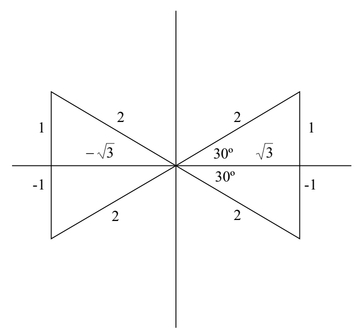
Algebra for Engineers

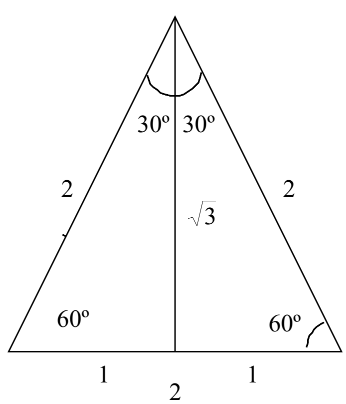


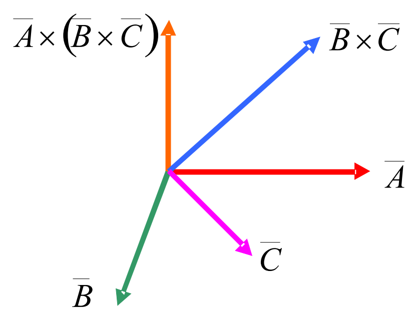
$$x^4 + x^3 - 2x^2 - 6x - 4 = 0$$

$$\left(\begin{array}{ccc|c} 5 & 4 & 3 & 6 \\ 2 & 3 & 2 & 3 \\ \vec{6} & 7 & 4 & 7 \end{array}\right)$$









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Algebra for Engineers

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This book is aimed at students in the first semester of our faculty which can provide support to the textbook which is currently used in the FIME.

CONTENTS

Complex Numbers

Companion student:

In order to understand better east Subject it is necessary that you know to use General Formula, The Theorem of the Binomial, The Theorem of Pythagoras and The trigonometrically Functions.

DEFINITION OF THE IMAGINARY UNIT

In our treaty of fractional exponents we excluded **the ROOTS** of EVEN **order** of the negative numbers. But the roots of even order, especially **the SQUARE ROOTS,** of the Negative **numbers** are very important in mathematics. Its use has contributed to the development of a great area of the mathematics, great part of which has vital application in engineering and physical sciences.

The equations that cannot be solved and many are many the problems that cannot be investigated in the system of the real numbers. The equation

$$a^2 + 1 = 0$$

for example, it has single solution if and if

$$a^2 = -1.$$

But as the square of any real number is not negative then and by consequence the equation does not have Real solutions. In order to obtain numbers whose squares are negative, we must extend our numerical system to include a new set of numbers.

The radicals of EVEN **order** with being (-) negative will be solved in the usual form adding to the result the letter i that means imaginary (because they do not have real solution) and it is defined of the following way:

$\sqrt{-1} = i$ imaginary unit

then like $\sqrt{-1} = i$

therefore $i^2 = -1$

and like $i^3 = i^2 \times i = -1 \times i = -i$

and like $i^4 = i^2 \times i^2 = -1 \times -1 = 1$

and like $i^5 = i^2 \times i^2 \times i = -1 \times -1 \times i = i$

etc.

Examples:

$$\sqrt{-4} = \pm 2i$$
$$\sqrt{-16x^2} = \pm 4xi$$
$$\sqrt{-11} = \pm \sqrt{11}i$$

The Complex Numbers have the canonical form $a+bi$, in where a it is the Real part and bi it is the imaginary part.

Like any number Real the Complex Numbers also have their negative and conjugated his, first is obtained to multiply he number complex by -1, and the second is obtained to change the sign of the Imaginary part.

We will try to explain the previous thing with the following examples:

Complex Numbers	**Negative**	**Conjugated**
$a + bi$	$-a - bi$	$a - bi$
$3 + 2i$	$-3 - 2i$	$3 - 2i$
$4 + 3i$	$-4 - 3i$	$4 - 3i$
$-2 - 5i$	$2 + 5i$	$-2 + 5i$
$3i$	$-3i$	$-3i$

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

SUM And REDUCES. The procedure to conduct these operations is similar that in the Real Numbers or is the Real part of a Complex Number Sum or Reduced á with the Real part of another Number Complex and the Imaginary part of a Complex Number Sum or Reduced a with the Imaginary part of another Complex Number.

The following example will give one more an ampler idea us of which previously I explain myself.

$$2+4i+3-5i-8+2i+12+3i=9+4i$$

MULTIPLICATION. We must remember that in Multiplication two basic operations take place, Multiplication and Sum, therefore when the operation takes place to multiply can be multiplied Real parts with Real parts or with Imaginary parts and vice versa and when it must carry out the operation to add will be as it were explained in the point No. 1.

The following example clarified the doubts.

$$\begin{array}{rl} & 2+3i \\ \times & \underline{3-4i} \\ & 6+9i \\ & \quad -8i-12i^2 \\ \hline & 6+1i-12i^2 \end{array}$$

but like $i^2=-1$ the answer is

$$6+i+12=18+i$$

DIVISION. When it must carry out division of two Complex Numbers, the process to follow is similar to which is carried out when rationalize Denominator, in other words the Complex Number that is going away to divide to be multiplied to him a as much to the dividend as to the splitter the conjugated one of the splitter and what is it will be the result.

Example : To divide $3+2i$ between $2+2i$

$$\frac{3+2i}{2+2i}\bullet\frac{2-2i}{2-2i}=\frac{6-2i-4i^2}{4-4i^2}=\frac{6-2i+4}{4+4}=\frac{10-2i}{8}$$

POWERS. In order to conduct this operation the indications will be followed that were explained in The Theorem of Binomial

Example: To elevate $2+3i$ to the fourth power

$$
\begin{aligned}
(2+3i)^4 &= (2)^4 + 4(2)^3(3i) + 6(2)^2(3i)^2 + 4(2)(3i)^3 + (3i)^4 \\
&= 16 + 96i + 216i^2 + 216i^3 + 81i^4 \\
&= 16 + 96i - 216 - 216i + 81 \\
&= -119 - 120i
\end{aligned}
$$

Note: To remember that $i^2 = -1$

$$i^3 = i^2 \times i = -1 \times i = -i$$

$$i^4 = i^2 \times i^2 = -1 \times -1 = 1$$

ROOTS. The operation to calculate roots of a Complex Number is as if we elevated that Complex Number to a Power Factionaries and therefore and in agreement with Theorem of Binomial the development tends to Infinite.

Example: To find the root fourth of $3+3i$

$$(3+3i)^{1/4} = (3)^{1/4} + \frac{1}{4}(3)^{-3/4}(3i) + \;\ldots\ldots\ldots.\infty$$

<u>Complex Numbers II</u>

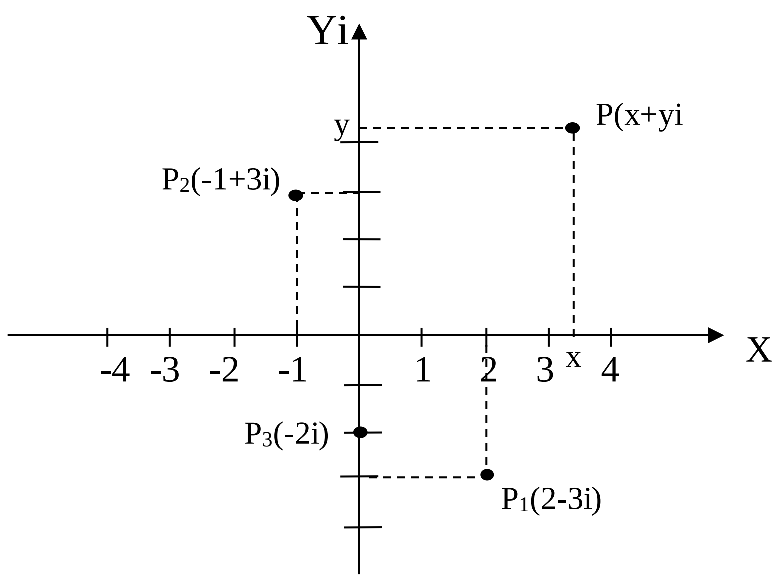
Companion student:

In order to understand better east Subject it is necessary that you know to use General Formula , the Theorem of the Binomial , the Theorem of Pythagoras and the Trigonometrically Functions .

RECTANGULAR FORM

I know to sight previously that the real numbers can imagine geometrically as points in an airline of equal way we try geometrically to represent he complex number assigning the real part of the complex number the axis X and to the imaginary part of the complex number the axis Y

Examples:

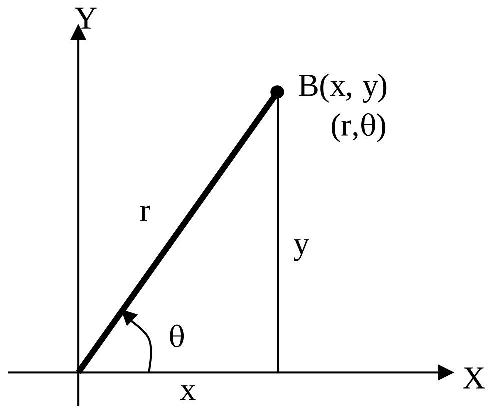


In agreement with the explained thing previously we can conclude that we can graphical any complex number in the axes.

IT FORMS POLAR

Polar Forma is Trigonometrically Form of the complex numbers which when carrying out algebraic operations with them have certain advantages on the rectangular form.

If we have a complex number $B(x+yi)$ located in the Y-axes, later if we drew up of the point B a perpendicular to the axis X and of the same point B we drew up a line to the origin would have left to a triangle rectangle:



Of where by Trigonometry Side Adjacent to Angle θ it is,

$$x = rCos\theta$$

And Side or position to Angle θ it is

$$y = rSen\theta$$

Therefore if his complex number that graphically is $x+yi$ and we equaled it to its corresponding one would have left:

$$x + yi = r(Cos\theta + iSen\theta)$$

In where the Left side of the equation is called Rectangular Form to him and alongside straight Polar Form is called to him, length r is called Module to him and to the angle θ Argument is called to him.

In order to clarify if we are going to change a complex number that this in Rectangular Form to Polar Forma we will have to calculate the Module (r) and the Argument (θ)

As it module of the polar form is the Hypotenuse of a Triangle Rectangle we will be able to calculate by means of Theorem of Pythagoras

$$r = \sqrt{x^2 + y^2} \quad \therefore \quad r \geq 0$$

and the Argument (angle) we will calculate it by Function TRIGONOMETRICAL TANGENT

$$Tg\,\theta = \frac{y}{x} \quad \therefore \quad x \neq 0$$

With the following example we will be able to clarify the doubts

To obtain Module, Argument and Polar Form of the following complex number $3+4i$

As it were already said previously the module will calculate with the Theorem of Pythagoras and the argument by the Tangent Trigonometrically Function .

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$Tg\,\theta = \frac{y}{x} = \frac{4}{3} = 1.3333 \quad \therefore \quad \theta = Tg^{-1} 1.3333 = 53^\circ 7' 48.37''$$

Therefore Polar Form is:

$$5\left(Cos 53^\circ 7' 48.37'' + iSen 53^\circ 7' 48.37''\right)$$

Of equal way if we had a complex number in Polar Form and wanted to change it single Rectangular Form it would be enough in multiplying he respectively module of the complex number by the Cosine and the Sine of the angle.

With the following example we will clarify the doubts.

To calculate Rectangular Form of the following complex number that this in Polar Form $4\left(Cos60^{\circ} + iSen60^{\circ}\right)$

$$4\left(Cos60^{\circ} + iSen60^{\circ}\right) = 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2 + 2\sqrt{3}i$$

OPERATIONS FUNDAMENTALES (in polar form)

It is of vital importance of knowing as the complex numbers that are in the polar form, by such reason are used we describe next the procedure of their use.

MULTIPLICATION. In order to conduct this operation To Multiply the Modules and the Arguments are added.

Example: To calculate the product of the following complex numbers

$$\begin{aligned} &3\left(Cos30^{\circ} + iSen30^{\circ}\right) \times 4\left(Cos40^{\circ} + iSen40^{\circ}\right) \\ &= 12\left(Cos70^{\circ} + iSen70^{\circ}\right) \end{aligned}$$

DIVISION. In order to conduct this operation the Modules will divide and I know Reduced the Arguments (to the argument of above the argument of down is reduced to him).

Example: To calculate the quotient of the following complex numbers

$$\frac{15\left(Cos75^{\circ} + iSen75^{\circ}\right)}{5\left(Cos45^{\circ} + iSen45^{\circ}\right)} = 3\left(Cos30^{\circ} + iSen30^{\circ}\right)$$

Note: It is necessary to have well-taken care of when carrying out the subtraction of the arguments, because if the argument of the complex number of above is smaller than the argument of the complex number of down, the subtraction will be NEGATIVE and we must remember that the arguments (angles) are always POSITIVE therefore will have to him to add 360°.

Example: To calculate quotient of the following complex numbers

$$\frac{15\left(Cos35^\circ + Sen35^\circ\right)}{3\left(Cos75^\circ + iSen75^\circ\right)} = 5\left(Cos - 40^\circ + iSen - 40^\circ\right)$$
$$= 5\left(Cos320^\circ + iSen320^\circ\right)$$

POWERS. Since the involution is a special case of the multiplication, and remembering that the operation to multiply be that they are multiplied the modules and the arguments are added, and by consequence we have if two complex numbers are equal and they are multiplied, its product would be.

$$\left[r\left(Cos\theta + iSen\theta\right)\right]^2 = r^2\left(Cos2\theta + iSen2\theta\right)$$

Like also:

$$\left[r\left(Cos\theta + Sen\theta\right)\right]^3 = r^3\left(Cos3\theta + iSen3\theta\right)$$

This makes us think that for any whole number and positive N we will have:

$$\left[r\left(Cos\theta + iSen\theta\right)\right]^N = r^N\left(CosN\theta + iSenN\theta\right)$$

Which is known him like the Theorem of MOIVRE?

Example: To elevate $\left[2\left(Cos30^\circ + iSen30^\circ\right)\right]$ **to the fifth power**

$$\left[2\left(Cos30^\circ + iSen30^\circ\right)\right]^5 = 32\left(Cos150^\circ + iSen150^\circ\right)$$

ROOTS. If to the Theorem of MOIVRE one rises to a fractional power we would have left.

$$\left[r\left(Cos\theta + iSen\theta\right)\right]^{1/N} = r^{1/N}\left(Cos\frac{\theta}{N} + iSen\frac{\theta}{N}\right)$$

And as to any angle him a multiple of 360 can be added ˚ it would be that:

$$[r(Cos\theta + iSen\theta)]^{1/N} = \left[r^{1/N} \left(Cos\frac{\theta + k \bullet 360^{\circ}}{N} + iSen\frac{\theta + K \bullet 360^{\circ}}{N} \right) \right]$$

We will make an example to clarify doubts.

To calculate the three cubical roots of: $8(Cos60^{\circ} + iSen60^{\circ})$

Applying the MOIVRE theorem to find the roots we would have left

$$[8(Cos60^{\circ} + iSen60^{\circ})]^{1/3} = \left[8^{1/3} \left(Cos\frac{60^{\circ} + k \bullet 360^{\circ}}{3} + iSen\frac{60^{\circ} + K \bullet 360^{\circ}}{3} \right) \right]$$

When the $\boldsymbol{k = 0}$, $2(Cos20^{\circ} + iSen20^{\circ})$

When the $\boldsymbol{k = 1}$, $2(Cos140^{\circ} + iSen140^{\circ})$

When the $\boldsymbol{k = 2}$, $2(Cos260^{\circ} + iSen260^{\circ})$

That the three roots

Exercises

Change the complex number given to its negative and its conjugate.

Complex Numbers $a+bi$	Negative $-a-bi$	Conjugated $a-bi$
$-2+i$		
$5-2i$		
$-3-4i$		
$7i$		
3		

II solve the operations of complex numbers and express the answer in rectangular form.

1) $(3-2i)+(7+5i)-(5-2i)$

2) $(-5-i)-(2+3i)+(4-2i)$

3) $3i-(7+i)-(-3-2i)$

4) $(3+i)\times(1-2i)$

5) $\left(1-i\sqrt{3}\right)\times\left(1+i\sqrt{3}\right)$

6) $(2+i)\times(-2+i)$

7) $\frac{4-2i}{1+3i}$

8) $\frac{-2+i\sqrt{2}}{3-i\sqrt{2}}$

9) $\frac{\sqrt{2}-i\sqrt{3}}{\sqrt{3}+i\sqrt{2}}$

10) $(3-i)^3$

11) $(2+2i)^4$

12) $\left(1-i\sqrt{2}\right)^3$

III Transform to polar form of complex number.

1) $8-6i$

2) $-3+6i$

3) $9+3i$

4) $-2+2i$

IV Transform to the rectangular form complex number.

1) $2(Cos135° + iSen135°)$

2) $4(Cos210° + iSen210°)$

3) $6(Cos300° + iSen300°)$

V Solve the operations of complex numbers in polar form.

1) $2(Cos70° + iSen70°) \times 5(Cos40° + iSen40°)$

2) $4(Cos60° + iSen60°) \times 6(Cos90° + iSen90°)$

3) $6(Cos70° + iSen70°) \times (Cos200° + iSen200°)$

4) $\frac{20(Cos95°+iSen95°)}{4(Cos30°+iSen30°)}$

5) $\frac{64(Cos220°+iSen220°)}{32(Cos40°+iSen40°)}$

6) $\frac{15(Cos280°+iSen280°)}{5(Cos70°+iSen70°)}$

7) $\left[\sqrt{2}(Cos225° + iSen225°)\right]^3$

8) $\left[\sqrt{3}(Cos60° + iSen60°)\right]^4$

9) $[2(Cos150° + iSen150°)]^5$

10) $[81(Cos200° + iSen200°)]^{\frac{1}{4}}$

11) $[64(Cos135° + iSen135°)]^{\frac{1}{3}}$

12 $[8(Cos90° + iSen90°)]^{\frac{1}{3}}$

SOLUTIONS

I

Complex Numbers $a+bi$	Negative $-a-bi$	Conjugated $a-bi$
$-2+i$	$2-i$	$-2-i$
$5-2i$	$-5+2i$	$5+2i$
$-3-4i$	$3+4i$	$-3+4i$
$7i$	$-7i$	$-7i$
3	-3	

II

1) $5+5i$

2) $-3-6i$

3) $-4+4i$

4) $5-5i$

5) 4

6) -5

7) $-\frac{1}{5}-\frac{7}{5}i$

8) $-\frac{8}{11}+\frac{\sqrt{2}}{11}i$

9) $-i$

10) $18-26i$

11) -64

12) $-5-i\sqrt{2}$

III

1 $)10(Cos323.13° + iSen323.13°)$

2) $)3\sqrt{5}(Cos243.43° + iSen243.43°)$

3) $)3\sqrt{10}(Cos18.43° + iSen18.43°)$

4) $)2\sqrt{2}(Cos135° + iSen135°)$

IV-

1) $-\sqrt{2}+i\sqrt{2}$

2) $-2\sqrt{3}-2i$

3) $3-3i\sqrt{3}$

V

1) $10(Cos110° + iSen110°)$

2) $24(Cos150° + iSen150°)$

3) $6(Cos270° + iSen270°)$

4) $5(Cos65° + iSen65°)$

5) $2(Cos180° + iSen180°)$

6) $3(Cos\ 210° + iSen210°)$

7) $2\sqrt{2}(Cos\ 315° + iSen315°)$

8) $9(Cos\ 240° + iSen240°)$

9) $32(Cos\ 30° + iSen30°)$

10) $k = 0$ $\quad 3(Cos50° + iSen50°)$

$k = 1$ $\quad 3(Cos140° + iSen140°)$

$k = 2$ $\quad 3(Cos230° + iSen230°)$

$k = 3$ $\quad 3(Cos320° + iSen320°)$

11) $k = 0$ $\quad 4(Cos45° + iSen45°)$

$k = 1$ $\quad 4(Cos165° + iSen165°)$

$k = 2$ $\quad 4(Cos285° + iSen285°)$

12) $k = 0$ $\quad 2(Cos30° + iSen30°)$

$k = 1$ $\quad 2(Cos150° + iSen150°)$

$k = 2$ $\quad 2(Cos270° + iSen270°)$

Theory of equations

Polynomial's Functions

The objective of this Subject is that the student learns to work the Polynomials and to obtain his Zeros Rationales, Irrationals and Complexes, as well as to learn to Factorize any Polynomial expressing it like the Product of Linear and Quadratic Factors.

A Polynomials Function of degree will be considered Whole m that it has the form:

$$f(x) = a_m x^n + a_{m-1} x^{n-1} + a_{m-2} x^{n-2} + ... + a_2 x^2 + a_1 x^1 + a_0$$

In which the Real Coefficients $a_m, a_{m-1}, a_{m-2}, ...a_1, a_0$ they are Constant, in where a_m it is the Main Coefficient and a_0 it is the Coefficient of the Independent term of x.

The terms of the Function are formed in descending order of the Powers of $"X"$ and I know will be analyzing stops $f(x) = 0$ since the Zero of the Polynomial $f(x)$ it is the Solution of the Whole Equation.

They exist You formulate to find the Solutions of Equations Polynomial's equal Grade to 3 and equal 4 to but its solution is very laborious to obtain and it is not very practices, in addition in superior treaties of Algebra also to I am demonstrated that the Equations of Degree equal or greater than 5 do not have Algebraic solution for this reason will set out another Method to determine them.

Theorems of the Remainder and the Factor

Theorem of the Remainder

A very useful form to determine the Zeros of a Polynomial $f(x)$ it is the Theorem of the Remainder, which we are going to introduce next.

If we carried out the Algebraic Division of a Polynomial

$$f(x) = 3x^3 - 4x^2 - 3x - 4$$

Between $x - 2$ where -2 it is an Independent number of x we would have left:

$$
\begin{array}{r|l}
 & 3x^2 + 2x + 1 \\ \hline
X-2 & 3x^3 - 4x^2 - 3x - 4 \\
 & \underline{-3x^3 + 6x^2} \\
 & \qquad 2x^2 - 3x \\
 & \qquad \underline{-2x^2 + 4x} \\
 & \qquad\qquad x - 4 \\
 & \qquad\qquad \underline{-x + 2} \\
 & \qquad\qquad\quad -2
\end{array}
$$

In where the quotient is $3x^2 + 2x + 1$ and the Remainder is -2

The Polynomial, then, can be expressed like:

$$3x^3 - 4x^2 - 3x - 4 = (x-2)(3x^2 + 2x + 1) - 2$$

Next, if we calculated $f(2)$ in the previous example, (if we remembered $f(2)$ it is obtained replacing 2 by x in the Function)

$$f(2) = 3x^3 - 4x^2 - 3x - 4 = 3(2)^3 - 4(2)^2 - 3(x) - 4 = -2$$

We can observe that the value of $f(2)$ he is equal to the value of the remainder that obtained in the Algebraic Division this could indicate that it is a coincidence nevertheless if the same procedure with several divisions takes place of $f(x)$ between different $x - r$ it would be possible to be verified that in all the cases that $f(r)$ he is equal to the remainder R which constitutes the foundation of the Theorem of the Remainder

THEOREM OF THE REMAINDER

If the Polynomial $f(x)$ Is divided between the Binomial $x - r$ where r it is a Real Number, the Remainder is equal a $f(r)$

Theorem of the Factor

Taking as it bases the Theorem of the Remainder, the statement of this Theorem can be established that will be very useful to determine the Factors to us of a Polynomial.

It is important to remember that when carrying out an Algebraic Division, if the Division is Exact the Remainder is equal to Zero.

THEOREM OF THE FACTOR

If r it is a root of $f(x)=0$ then $x-r$ it is a factor of $f(x)$.

With the previous thing, it is possible to be made notice the importance of knowing the value the remainder, since if this he is equal to zero, is going to us to indicate that Factors are had, and that with them can be determined the zeros of the polynomial.

It is of vital importance that you know to use **the Synthetic Division**.

Graphs of Polynomial's Functions

In this section I know side like graphical the polynomial's functions, that are of the form:

$$f(x)=a_m x^n + a_{m-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

Where each a_i it is a Real number, one has already studied previously that if $f(x)$ it has degree $n=1$ then the graph of the function is an airline, whereas if $f(x)$ it is of degree $n=2$ then the graph of $f(x)$ it is a parabola. Now the graphs of polynomials were considered $f(x)$ of degree $n \geq 3$.

FUNDAMENTAL THEOREM OF ALGEBRA

- All whole Equation $f(x)=0$ (zero) have at least one Root either Real or Complex
- A whole Equation $f(x)=0$ of Degree n it has exactly n Roots (zeros)

Multiplicity: If the factor $x-r$ it happens k times, one says that r he is a Zero of multiplicity k.

Given the Roots of a Polynomial to find the Polynomial

If we have a Polynomial

$$x^2 - 9x + 20 = 0$$

and the Factorize we have left;

$$(x-4)(x-5)=0$$
$$\therefore$$
$$(x-4)=0,(x-5)=0$$
$$x=4$$
$$y$$
$$x=5$$

Now if some Roots of a Polynomial occur us $2,-3$ and 5 to find the Polynomial:

Solution: it will be come in like the previous example but in Reversal

$$x=2 \qquad x=-3 \qquad x=5$$

$$x-2=0 \qquad x+3=0 \qquad x-5=0$$

$$(x-2)(x+3)(x-5)=0$$

$$x^3-4x^2-11x+30=0$$

Theory of equations II

Zeros Conjugated Complexes

Remember that the conjugated one of $a+i$ it is $a-bi$

Theorem about the Conjugated Complexes

If the Complex Number $a+bi$ (where b it is different from 0) he is a Zero (Root) of the Polynomial with Real Coefficients, then Conjugated his $a-bi$ also he is a Zero (Root) of the Polynomial.

THEOREM

If the Irrational Number $a+\sqrt{b}$ he is a Zero (Root) of a Polynomial that has Rational Coefficients, then the Irrational Number $a-\sqrt{b}$ also a Zero (Root) del Polynomial

Levels

Often he is useful to be able to determine an interval that contains all the Zeros Real ones of a Polynomial Function. All Number that is Greater that/o Equal to Zero Greater of a Polynomial Function receives the name of Superior Level of the Zeros, equal way, all Number that is Equal a/o Smaller than the Zero but Small from a Polynomial Function it receives the name of Inferior Level of the Zeros.

In the theorem that follows it uses **the Synthetic Division** in order to determine Superior and Inferior Levels and Theorem of the Levels is called to him.

THEOREM OF THE LEVELS

Suppose that $f(x)$ it is a Polynomial of Real Coefficients and that its first Coefficient is Positive. If we used **Synthetic Division** in order to divide $f(x)$ between $x-r$

- If $r \geq 0$ and all the Terms of the third Line of **the Synthetic Division** they are Positive or 0, then r it is a Superior Level of Zeros

- If $r \leq 0$ and all the Terms of the third Line of **Synthetic Division** they have Alternate Signs (+, -), then r it is an Inferior Level of the Zeros

Rule of the Signs of Discarding

The following Rule allows determining the Maximum Number of Zeros Positives and Negatives of a Polynomial $f(x)$ with Real Coefficients.

If the Terms of a Polynomial are written normally in Descendent Powers of the Variable, it says that it happens a variation of signs when the Signs of two Consecutive Terms are Different.

Rule of the Signs of Discarding

Be $f(x)$ a Polynomial that has Real Coefficients and an Independent term different from Zero,

- The number of Positive Roots of $f(x)=0$ he is equal to the Number of Variations of sign of $f(x)$ or she is Minor who this number in an Even Amount.
- The number of Negative Roots of $f(x)=0$ he is equal to the Number of Variations of sign of $f(-x)$ or she is Minor who this number in an Even Amount.

We will see an example to clarify doubts

$$f(x)=2x^5+3x^4-2x^3+4x^2-2x-5$$

As it is possible to be observed between Second and Third, between Third and Fourth and the Fourth and Fifth Term there are Variations of Sign, therefore it has 3 Variations,

Now if we calculated $f(-x)$

$$f(-x)=-2x^5+3x^4+2x^3+4x^2-2x-5$$

As it is possible to be observed between First and Second and between the Fourth and Fifth Term there are Variations of Sign, therefore it has 2 Variations, therefore the Rule of the Signs of Discarding we would have left:

The Roots can be:

Positive	Negatives	Imaginary
3	2	0
1	2	2
3	0	2
1	0	4

Zeros Rationales

If a Polynomial Function has one more or Zeros Rationales, the work that are due to make to determine the other Zeros I know side substantially reduced if First the Zeros Rationales are located. The identification of a Zero Rational is a process Error and Success (rough estimates), being the number of Attempts limited by the Theorem that follows.

THEOREM OF THE ZEROS RATIONALS

Suppose that the Coefficients of a Polynomial Function

$$f(x) = a_m x^n + a_{m-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

They are whole and **the P/q Relation** it is a zero rational in its Minimum Expression. In where the numerator p it is a Factor of the Constant Term and the denominator q it is a Factor of the Main Coefficient

With the following example the doubts will be clarified:

$$4x^4 + 2x^3 - 4x^2 + 2x - 6 = 0$$

$$\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2, \pm 4} =$$

$$\therefore$$

$$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{3}{2}, \pm \frac{3}{4}$$

These numbers are the possible Rational Roots of the Polynomial

Factorize of Polynomial

All polynomial with Real Coefficients can be written like the Product of Linear and Quadratic Factors (irreducible in the field of the real numbers), having each factor Real Coefficients.

For example: given the Polynomial $x^3 - 7x^2 + 17x - 15 = 0$ to express it like a Product of Linear and Quadratic Factors.

Solution: if we used the relation

$$\frac{p}{q} = \frac{\pm 1, \pm 3, \pm 5, \pm 15}{\pm 1} = \pm 1, \pm 3, \pm 5, \pm 15$$

and if we calculated Error and Success with Synthetic Division I know side that the only value that is Root of the given Polynomial is 3 therefore $x-3$ it is the Linear factor and the Quotient of the Division $x^2 - 4x + 4$ it is the Quadratic Factor therefore

$$x^3 - 7x^2 + 17x - 15 = (x-3)(x^2 - 4x + 4)$$

Theory of equations III

Companion in this subject who is wanted is that you learn to find the Roots (zeros) of a Whole Equation of Degree N; the process that we are going to use is the one of Rough estimates.

In this Course We will limit us single those Equations in which all the Coefficients of the same one are REAL.

In first instance we will try to later find the Roots Real (rational or irrational) and the Imaginary ones which will be by means of General Formula.

The Steps to follow to find the Roots are the following ones:

1. To determine by means of Rule of the Signs of Discarding and to where it is possible the number of Positive Roots, Negative and Imaginary that the Equation Can have.
2. To find by means of the Relation P in q $\left(\frac{P}{q}\right)$ (P is factors of respectively finish independent and to q the factors of the main coefficient) the possible Rational Roots.
3. To prove the possible Roots by means of Synthetic Division :
4. The Roots will be those that when calculating them by means of Synthetic Division the Remainder is ZERO.
5. Whenever you obtain a Root you must separate it, and you will use the Reduced Equation to look for the following one.
6. You will continue east process until the Reduced Equation is of Degree 2 which will be able to be solved by Factorize or by General Formula.

Example: To find the Roots of the following Equation.

$$x^4 + x^3 - 2x^2 - 6x - 4 = 0$$

One variation

· Rule of the Signs of Discarding

+	-	i
1	3	0
1	1	2

· To calculate the Relation $\frac{p}{q}$

$$\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1} = \pm 1, \pm 2, \pm 4$$

·To prove each one by these possible Roots by means of Synthetic Division or it is to calculate $1, 2, 4, -1, -2\ and - 4$ As you can be given account if you already calculated 2 and -1 is Roots, since the Remainder in both is Zero, Calculates the first 2 later uses the Reduced equation (third line) that is of Degree 3 to calculate number -1, and you will obtain the following Reduced Equation that is of Degree 2.

The missing Roots you will calculate them by means of it General Formula and they are $-1 \pm i$

- Tips:

1. When for some reason you cannot find the Roots of one Equation graphically for thus this way seeing where it touches graph X-axis.
2. If you already found some Roots but Not All It repeats in Synthetic Division those Roots that already you found (can that there are Repeated Roots), if even so you continue Needing Roots remembers that they can be Imaginary Roots (which you found them by means of General Formula) or Irrational Roots (they will be possible to be found by the Method of Horner)

EXERCISES

I Build the polynomial equation that has the roots:

1) $2, -2 \; and \; 3$

2) $\sqrt{3}, -\sqrt{3} \; and \; 1$

3) $-1, 2, -3 \; and \; 3$

4) $1, 1, -2 \; and - 1$

II Find all possible information about the nature of the roots of the given equation, through the rule of Descartes and determine the possible rational roots.

1) $2x^5 - 3x^4 + 2x^3 - 3x^2 - 24x + 36$

2) $2x^4 - 2x^3 - 29x^2 - 28x + 12$

3) $12x^5 - 6x^4 + 60x^3 - 30x^2 + 72x - 36$

III Express the polynomial factors

1) $6x^3 - 7x^2 - 29x - 12$

2) $2x^3 - 3x^2 - 6x + 9$

3) $x^3 + 6x + 20$

4) $2x^4 - 11x^3 + 18x^2 - 4x - 8$

5) $9x^4 - 12x^3 + 40x^2 - 48x + 16$

ANSEWERS

I

1) $x^3 - 3x^2 - 4x + 12$

2) $x^3 - x^2 - 3x + 3$

3) $x^4 - x^3 - 11x^2 + 9x + 18$

4) $x^4 + x^3 - 3x^2 - x + 2$

II

1)

$$\frac{p}{q} = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3, \pm 4, \pm\frac{9}{2}, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$$

Positive	Negatives	Imaginary
4	1	0
2	1	2
0	1	4

2)

$$\frac{p}{q} = \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm\frac{4}{3}, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$

Positive	Negatives	Imaginary
2	2	0
2	0	2
0	2	2
0	0	4

3)
$\frac{p}{q} = \pm\frac{1}{12}, \pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{3}{4}, \pm 1, \pm\frac{4}{3}, \pm\frac{3}{2}, \pm 2, \pm\frac{9}{4}, \pm 3, \pm 4, \pm\frac{9}{2}, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$

Positive	Negatives	Imaginary
5	0	0
3	0	2
1	0	4

III

1) $(x-3)(2x+1)(3x+4)$

2) $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)(2x-3)$

3) $(x^2-2x+10)(x+2)$

4) $(2x+1)(x-2)^3$

5) $(x^2+4)(3x-2)^2$

DETERMINANTS OF SECOND And THIRD ORDER

Definition.

If To it is a square matrix of order two, to this a number can be assigned to him that receives the DETERMINANT name and can represent by a letter anyone (A).

DETERMINANTS OF SECOND ORDER

Definition of the Value.

The value of a determinant of order two, is defined as the product of the elements of the main diagonal (↘) minus the product of the elements of the other diagonal (↗)

This is:

$$\text{If} \quad A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Example. Find the value of the indicated determinants:

$$C_2 = \begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} = (2)(2) - (4)(2) = 4 - 8 = -4$$

$$D_2 = \begin{vmatrix} 6 & 2 \\ -3 & 3 \end{vmatrix} = (6)(3) - (-3)(2) = 18 + 6 = 24$$

DETERMINANTS OF THIRD ORDER

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 - c_1 b_2 a_3 + b_1 c_2 a_3 - a_1 c_2 b_3 + c_1 a_2 b_3 - b_1 a_2 c_3$$

As the solution can be observed of Δ_3 it is not; "the main diagonal, except the other diagonal" since they appear six terms here that must somehow be justified their value.

In order to visualize the solution of a method I practice, and "exclusive" for determinants of order three the following thing can be done.

Rain method

To add the first two columns to the right of the determinant and to draw up to diagonals uniting three elements in the direction of the main diagonal will give the terms us with positive sign, and the diagonals that "cross" the main diagonal uniting three elements will give the terms us with negative sign:

$$\Delta_3 = \left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right| =$$

$$a_1b_2c_3 \quad + \quad b_1c_2a_3 \quad + \quad c_1a_2b_3 \quad - \quad c_1b_2a_3 \quad - \quad a_1c_2b_3 \quad - \quad b_1a_2c_3$$

REMEMBER. This procedure is exclusive stops Δ_3 .

Example. Find the value of the determinant.

$$\Delta_3 = \begin{vmatrix} 1 & 6 & -1 \\ 2 & -2 & 6 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\Delta_3 = \left|\begin{array}{ccc|cc} 1 & 6 & -1 & 1 & 6 \\ 2 & -2 & 6 & 2 & -2 \\ 3 & 4 & 2 & 3 & 4 \end{array}\right| =$$

$$= (1)(-2)(2) + (6)(6)(3) + (-1)(2)(4) - (3)(-2)(-1) - (4)(6)(1) - (2)(2)(6) =$$

$$= -4 + 108 - 8 - 6 - 24 - 24$$

$$\therefore$$

$$\Delta_3 = 42$$

DETERMINANTS OF ORDER N

SOLUTION BY COFACTORES

The student will wonder itself if a unique method exists that solves determinants of any order, the answer is affirmative and its demonstration starting off of the general solution will occur of Δ_3 .

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - c_1b_2a_3 + b_1c_2a_3 - a_1c_2b_3 + c_1a_2b_3 - b_1a_2c_3$$

Removing common factor and grouping (observing the first row)

$$\Delta_3 = a_1(b_2c_3 - c_2b_3) \quad + \quad b_1(c_2a_3 - a_2c_3) \quad + \quad c_1(a_2b_3 - b_2a_3)$$

Changing sign to the second term

$$\Delta_3 = a_1(b_2c_3 - c_2b_3) \quad - \quad b_1(a_2c_3 - c_2a_3) \quad + \quad c_1(a_2b_3 - b_2a_3)$$

What this between parenthesis is written with determinants of second order

(Δ_2)

$$\Delta_3 = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

It is observed that the determinants that accompany the elements to $_1$b $_1$c $_1$obtains when eliminating the row and column to that they belong respectively, and that one of them has negative sign. These determinants receive the name of COFACTOR of an element of a determinant being their definition as it follows:

Definition.

COFACTOR of an element of a determinant is called to the determinant of inferior immediate order that obtains when suppressing the row and column to that this element belongs and that in addition has positive or negative sign.

In order to justify the sign of the cofactor of the element, it is possible to be thought about two forms.

1. It will have positive **sign** if the position of the element as far as the sum of row and column is even **and** negative **number** if the sum gives **odd number** .

2. The sign of the cofactor of the element of a determinant will have positive or negative sign according to the following "table" of signs.

$$\Delta\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The value of any determinant of order n, is equal to one it adds algebraic of **n** terms, each one of which each element of any row or column by its corresponding COFACTOR forms when multiplying.

Example. To calculate the value of the Determinant of the previous example using the Method of Cofactors'

a): taking as row bases the elements of 1er

$$\Delta_3 = \begin{vmatrix} 1 & 6 & -1 \\ 2 & -2 & 6 \\ 3 & 4 & 2 \end{vmatrix}$$

Solution.

a) Base to 1er row.

$$\Delta_3 = 1\begin{vmatrix} -2 & 6 \\ 4 & 2 \end{vmatrix} - (6)\begin{vmatrix} 2 & 6 \\ 3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -2 \\ 3 & 4 \end{vmatrix}$$

$$\Delta_3 = 1(-4-24) - 6(4-18) + (-1)(8+6)$$

$$\Delta_3 = -28 + 84 - 14 = 42$$

$$\therefore$$

$$\Delta_3 = 42$$

This method of solution "Complicated" when it is applied to Determinants of Superior Order.

This problem can be avoided if we know the Properties the Determinants to combine them with the solution by cofactors.

determinants II

SOLUTION OF LINEAR EQUATIONS BY MEANS OF DETERMINANTS

Given a System of Linear Equations with two incognitos to find its solution.

$$\begin{array}{llllll} 1 & a_1x & + & b_1y & = & k_1 \\ 2 & a_2x & + & b_2y & = & k_2 \end{array}$$

As a System were said previously in the subject of Matrices Linear Equations can be nize as they are the Method Elimination by Sum and Reduces the one of Substitution the will solve it by the Rule of Cramer

CRAMER RULE

As the solution of a system of two Equations with two Incognitos were seen previously this given by:

$$x = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2} \text{ and } y = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}$$

Solution that can be written with relation of Determinants like

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

And in simplified form:

$$x = \frac{\Delta x}{\Delta s} \qquad y = \frac{\Delta y}{\Delta s}$$

in where

$\Delta_s = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ it is the delta of the System and it is formed by the Coefficients,

$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ it is the delta "x" and this formed by the Constants (in the column of "x")

and by the Coefficients of" and " and

$\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ that it is the delta "and" and is formed by the Coefficients of "x" and by the Constants (in the column of "and")

The Rule of Cramer can be generalized to solve Systems of Linear Equations of n incognito equations with n, being written of the following form.

$$x = \frac{\Delta x}{\Delta s} \quad , \qquad y = \frac{\Delta y}{\Delta s} \quad , \qquad z = \frac{\Delta x}{\Delta s} \quad . \quad . \quad . \quad . \quad i = \frac{\Delta i}{\Delta s} \qquad etc.$$

Where:

i It represents anyone of the n incognito

Δs It is the determinant of the coefficients of the incognitos with

ordered equations of the system of equations.

Δi It is the determinant that forms when it replaces the terms

constants on the column of the incognito i in Δ_s

Example 1. To solve the Systems of Linear Equations given using the Rule of Cramer.

$$\begin{array}{ccccccc} x & + & 2y & - & z & = & 3 \\ 2x & - & y & + & z & = & 7 \\ 2x & + & y & - & 4z & = & -1 \end{array}$$

Solution: We will make four passages in the solution of a System of Linear Equations that are the following ones:

The delta of the System calculates (Δ s) formed by the Coefficients of the system of linear equations (by any method)

$$\Delta s = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 2 & 1 & -4 \end{vmatrix}$$

$$\Delta s = 1\begin{vmatrix} -1 & 1 \\ 1 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} + (2)\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}$$

$$\Delta s = 1(4-1) - 2(-8+1) + 2(2-1)$$

$$\Delta s = 3 + 14 + 2 = 19$$

$$\Delta s = 19$$

Conclusion: like $\Delta s \neq 0$ there is unique solution.

We calculated the deltas now: $\Delta x, \Delta y$ and Δz staying formed as it follows:

$$\Delta x = \begin{vmatrix} 3^{*} & 2 & -1 \\ 7 & -1 & 1 \\ -1 & 1 & -4 \end{vmatrix} \qquad \Delta y = \begin{vmatrix} 1 & 3^{*} & -1 \\ 2 & 7 & 1 \\ 2 & -1 & -4 \end{vmatrix} \qquad \Delta z = \begin{vmatrix} 1 & 2 & 3^{*} \\ 2 & -1 & 7 \\ 2 & 1 & -1 \end{vmatrix}$$

Observe as the constant terms occupy the column of the incognito to calculate (noticeable with asterisk) being:

$$\Delta x = 57 \qquad \Delta y = 19 \qquad \Delta z = 38$$

The calculations are left like exercise the student.

The incognitos calculate x, y and z using the Cramer rule.

$$x = \frac{\Delta x}{\Delta s} \qquad y = \frac{\Delta y}{\Delta s} \qquad z = \frac{\Delta z}{\Delta s}$$
$$x = \frac{57}{19} \qquad y = \frac{19}{19} \qquad z = \frac{38}{19}$$
$$x = 3 \qquad y = 1 \qquad z = 2$$

The solution by substitution in anyone of the given equations is verified.

$$\begin{array}{ccccccc} x & + & 2y & - & z & = & 3 \\ 3 & + & 2(1) & - & 2 & = & 3 \\ 3 & + & 2 & - & 2 & = & 3 \end{array}$$

Example 2. To solve the system of equations given.

$$\begin{array}{ccccccccc} x & + & 2y & + & z & - & 2w & = & -2 \\ 3x & - & y & - & z & + & w & = & 3 \\ 2x & - & y & + & 2z & - & 4w & = & 1 \\ 4x & - & 3y & - & 2z & + & w & = & 3 \end{array}$$

Solution. Just as in the previous example One calculates Δs by any method

$$\Delta s = \begin{vmatrix} 1 & 2 & 1 & -2 \\ 3 & -1 & -1 & 1 \\ 2 & -1 & 2 & -4 \\ 4 & -3 & -2 & 1 \end{vmatrix} =$$

$$\Delta_s = 1\begin{vmatrix} -1 & -1 & 1 \\ -1 & 2 & -4 \\ -3 & -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 & -2 \\ -1 & 2 & -4 \\ -3 & -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ -3 & -2 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ -1 & 2 & -4 \end{vmatrix} =$$

$$\Delta_s = 1(-2+2-12+6+8-1) - 3(4+12-4-12-16+1)$$

$$+2(-2-4-3+6+4+1) - 4(8-1+4+2-4-4) =$$

$$\Delta_s = 1(1) - 3(-15) + 2(2) - 4(5)$$

$$\Delta_s = 30$$

like $\Delta s \neq 0$ then there is unique solution

We calculated the deltas $(\Delta x, \Delta y, \Delta z,$ and $\Delta w)$

Again these calculations are suppressed so that it is exercise for the student.

$$\Delta y = \begin{vmatrix} 1 & -2 & 1 & -2 \\ 3 & 3 & -1 & 1 \\ 2 & 1 & 2 & -4 \\ 4 & 3 & -2 & 1 \end{vmatrix} \Rightarrow \Delta y = -30$$

$$\Delta x = \begin{vmatrix} -2 & 2 & 1 & -2 \\ 3 & -1 & -1 & 1 \\ 1 & -1 & 2 & -4 \\ 3 & -3 & -2 & 1 \end{vmatrix} \Rightarrow \Delta x = 30$$

$$\Delta z = \begin{vmatrix} 1 & 2 & -2 & -2 \\ 3 & -1 & 3 & 1 \\ 2 & -1 & 1 & -4 \\ 4 & -3 & 3 & 1 \end{vmatrix} \Rightarrow \Delta z = 90$$

$$\Delta w=\begin{vmatrix}1 & 2 & 1 & -2\\ 3 & -1 & -1 & 3\\ 2 & -1 & 2 & 1\\ 4 & -3 & -2 & 3\end{vmatrix}\Rightarrow \Delta w=60$$

We calculated the values of the variables using the Cramer rule.

$$\begin{array}{llll} x=\dfrac{\Delta x}{\Delta s} & y=\dfrac{\Delta y}{\Delta s} & z=\dfrac{\Delta z}{\Delta s} & w=\dfrac{\Delta w}{\Delta s}\\ x=\dfrac{30}{30} & y=\dfrac{-30}{30} & z=\dfrac{90}{30} & w=\dfrac{60}{30}\\ x=1 & y=-1 & z=3 & w=2 \end{array}$$

We verified the solution by substitution in nobody equation

$$\begin{array}{rcccccccccc} & x & + & 2y & + & z & - & 2w & = & -2\\ 3) & 1 & - & 2 & + & 3 & - & 4 & = & -2\\ & & & & & & & -2 & = & -2 \end{array}$$

As I know can observe if systems of equations of superior order are solved the process he becomes tedious, and it generates the fatigue mental if it is made by hand, reason why is recommendable the use of the Computer for these Systems of Linear Equations **(Mat-lab)** .

determinants III

DEFINITION And I CALCULATE OF INVERSE MATRIX

Within the definitions of Matrices that occurred at the beginning of the subject of Matrices, the definition of the Inverse Matrix was seen which we will remember.

Definition.

Be To a square matrix of order n, for which its determinant $|A|$ he is different from zero, and In their corresponding matrix identity, then it exists a square matrix of order n $\left(A^{-1}\right)$ so that: $A \bullet A_n^{-1} = I_n = A_n^{-1} \bullet A$ where A^{-1} it is the INVERSE MULTIPLICATIVA OF A. Which suggests a form of calculation of the Inverse Matrix which we will give next:

Be To a Square Matrix of Order 2 whose determinant $|A|$ he is different from zero.

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

Then its Inverse Matrix exists and this given by:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

Where A_1, A_2, B_1, B_2 they are COFACTORES of a_1, a_2, b_1, b_2.

Example. To calculate the Inverse Matrix of the given matrix if it exists.

$$B = \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix} \qquad B^{-1} = ?$$

Solution. First: we calculated $|B|$

$$|B| = \begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix} = 18 \text{ like } |B| \neq 0 \text{ if it exists } B^{-1}$$

Second: we replaced the elements of B by its respective COFACTORES having taken care of to place them in its place. (of rows to columns) to form the First Associate of B (B^{+})

$$B^{+} = \begin{pmatrix} |6| & -|4| \\ -|3| & |5| \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -3 & 5 \end{pmatrix}$$

Third: we wrote the Inverse Matrix by substitution in the expression:

$$B^{-1} = \frac{1}{|B|}(B^{+})$$

$$B^{-1} = \frac{1}{18}\begin{pmatrix} 6 & -4 \\ -3 & 5 \end{pmatrix}$$

where B^{+} she is the First Associate

It is possible to be verified that this "Method" of solution of I calculate of the Inverse Matrix is possible to be applied to matrices of order **n** following the same used procedure in those of order two and can be improved if a consisting of intermediate extra step writing the transposed matrix of the matrix is added to invest being left these steps as it follows:

· To calculate the Determinant of the Matrix $|A|$

· To write the Transposed Matrix (A^{t}

· To calculate the First Associate (A^{+})

· To write the Inverse Matrix (A^{-1} by substitution in:

$$A^{-1} = \frac{1}{|A|}(A^{+})$$

NOTE. - to write the Transposed Matrix of the Matrix, allows to replace each element of her and by her Cofactor to form the First Associate, avoiding to be changing of Rows (Lines) to Columns.

Example. Calculate the Inverse Matrix of C if it exists.

$$C = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 5 \end{pmatrix}$$

Solution: we will firstly calculate the value of the determinant $|C|$ by any method (smaller)

$$|C| = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 5 \end{vmatrix}$$

$$|C| = 3\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 4 & 5 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$|C| = 3(5-12) - 2(-5-8) + 3(-3-2)$$

$$|C| = 3(-7) - 2(-13) + 3(-5)$$

$$|C| = -21 + 26 - 15$$

$$|C| = -10$$

$$|C| \neq 0 \Rightarrow \text{It exists } C^{-1}$$

the transposed matrix of C is written (C^t), in where the lines of C are the columns in C^t

$$C^t = \begin{pmatrix} 3 & 2 & 3 \\ -1 & 1 & 4 \\ 2 & 3 & 5 \end{pmatrix}$$

the first associate of C calculates (C^+ replacing each element of C^t by his cofactor.

$$C^+ = \begin{pmatrix} \begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} -1 & 4 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 3 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -7 & 13 & -5 \\ -1 & 9 & -5 \\ 5 & -15 & 5 \end{pmatrix}$$

Inverse matrix C -1 by substitution is written in:

$$C^{-1} = \frac{1}{|C|}(C')$$

being of the following way:

$$C^{-1} = \frac{1}{-10}\begin{pmatrix} -7 & 13 & -5 \\ -1 & 9 & -5 \\ 5 & -15 & 5 \end{pmatrix}$$

Another form that exists for him I calculate of the Inverse Matrix consists of writing to the right of the Matrix who is desired to calculate his inverse one, the First corresponding

Identity forming a matrix of nxn2n and to make elementary transformations of line with the purpose of obtaining that the First Identity appears left alongside being left therefore the Matrix Inverse alongside straight.

Or to place to the right of the Matrix that is desired to find its Inverse one, the First Identity and to apply to the Algorithm Post (this method I know side later in Determinants 4).

Example: To calculate the Inverse C^{-1} of the following Matrix if it exists using the Method Post.

$$C = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 5 \end{pmatrix}$$

Solution. The Steps that are used to make this process find in AYUDÁNDOTE

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \ldots ETC \ldots \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{10} & \frac{-13}{10} & \frac{5}{10} \\ 0 & 1 & 0 & \frac{1}{10} & \frac{-9}{10} & \frac{5}{10} \\ 0 & 0 & 1 & \frac{-5}{10} & \frac{15}{10} & \frac{-5}{10} \end{array}\right)$$

final solution

APPLICATION OF THE INVERSE MATRIX

The main application of the inverse matrix it is in the Solution of Systems of Linear Equations by means of the Multiplication of two Matrices as it is demonstrated next.

Note: this application is the one that uses the computational program **Mat-lab** in order to solve a System of Linear Equations

Given a system of linear equations to express it with matrices.

(system of 3x3)

$$\begin{array}{ccccccc} a_1 x & + & b_1 y & + & c_1 z & = & k_1 \\ a_2 x & + & b_2 y & + & c_2 z & = & k_2 \\ a_3 x & + & b_3 y & + & c_3 z & = & k_3 \end{array}$$

Solution.

$$\begin{pmatrix} a_1 & b_1 & c_c \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

That they are respectively the Matrix of the Coefficients of the incognitos: (A), the Matrix of the Incognitos (x) and the Matrix of the constants (k) reason why in simplified form can be written of the following way:

$A \bullet X = k$ Matrix Equation.

If the inverse matrix of A (A^{-1}) it exists, we can multiply the previous equation by A^{-1} by the left, being:

$$A^{-1} \bullet A \bullet X = A^{-1} \bullet k$$

Like $A^{-1} \bullet A = I$ we have left:

$$I \bullet X = A^{-1} \bullet k$$

Also we know that $I \bullet X = X$

$$X = A^{-1} \bullet k$$

o

$$A^{-1} \bullet k = X$$

What it demonstrates to us that if we multiplied the Inverse Matrix of the Matrix of the Coefficients of a system of linear equations by the Matrix of the Constants the result it will be the Matrix of the Incognitos of the system; said of another form it will give the solution us of the System of Linear Equations, we see an example of this application.

Example. To solve the System of Linear Equations given using the Inverse Matrix of the Coefficients of the incognitos.

$$\begin{aligned} 2x + y - z &= 5 \\ 3x - 2y + 2z &= -3 \\ x - 3y - 3z &= -2 \end{aligned}$$

Solution. We calculated the value of the Determinant

$$A = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 2 \\ 1 & -3 & -3 \end{vmatrix} = 2\begin{vmatrix} -2 & 2 \\ -3 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ -3 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} =$$

$$A = 2(6+6) - 3(-3-3) + 1(2-2) = 24 + 18 + 0$$

$$\therefore$$

$$A = 42$$

Transposed his A^t it is:

$$A^t = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & -3 \\ -1 & 2 & -3 \end{pmatrix}$$

Its associate A^+ she is:

$$A^+ = \begin{pmatrix} +\begin{vmatrix} -2 & -3 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ -1 & -3 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ -2 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 12 & 6 & 0 \\ 11 & -5 & -7 \\ -7 & 7 & -7 \end{pmatrix}$$

Its inverse one To $^{-1}$ is:

$$A^{-1} = \frac{1}{|A|}(A^+)$$

$$A^{-1} = \frac{1}{42}\begin{pmatrix} 12 & 6 & 0 \\ 11 & -5 & -7 \\ -7 & 7 & -7 \end{pmatrix}$$

The solution of the system is expressed by:

$$A^{-1} \bullet K = X$$

$$\frac{1}{42}\begin{pmatrix} 12 & 6 & 0 \\ 11 & -5 & -7 \\ -7 & 7 & -7 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{60-18+0}{42} \\ \dfrac{55+15+14}{42} \\ \dfrac{-35-21+14}{42} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \therefore \quad x = 1 \quad y = 2 \quad z = -1$$

METHOD POST and ITS APPLICATIONS

We will present/display next the method of I calculate matrix well-known like the Method Post in its diverse applications such as:

· Solution De Determinants

· Solution De Sistemas De Linear Ecuaciones

· I calculate Of the First Associate

· I calculate of the Inverse Matrix Of a Given Matrix

· Etc.

This method unlike the method Gaussian Jordán presents/displays the advantage of which if when initiating the solution of any Matrix or Determining single east has Whole Numbers will only work with whole numbers and it does not give rise to the appearance of fractional elements facilitating the procedure in him I calculate manual. The method Post originally was called "Algorithm Post" because from the mathematical point of view it is an algorithm but from the numerical point of view it is a method.

This method was developed and applied by **the Ing. Rene Mario Montante Pardo Post in the Faculty of Mechanical and Electrical Engineering of the Independent University again Leon**

ALGORITHM POST

The algorithm Post is based on making transformations of the elements of the Determinants or the Matrices, to transform them to Determinants or equivalent Matrices.

These transformations "turn" on a called element "Pivot" (P) which can be in any Row (line) or Column. It formulates Post for the transformations of the elements is the following one:

$$N.E = \frac{(P)(E.A) - (E.C.F.P.)(E.C.CP.)}{(P.A)}$$

In where.

N.E.	New Element (or transformed element)
P.	Pivot
E.A.	Present element (or element to transform)
E.C.F.P.	Element Corresponding to the Row of the Pivot
E.C.C.P.	Element Corresponding to the Column of the pivot

P.A. Previous Pivot

ALGORITHM POST APPLIED To The SOLUTION OF DETERMINANTS

Next some steps will be detailed that will serve us to find the solution of Determinants

· Any element can be taken like "Pivot", except elements "zero", (to less of than it is the last transformation).

· The Pivot is chosen, It isolates the Row and the Column of the Pivot

· In the first transformation the "Previous Pivot" takes the value from one.

· The elements of row and column of the element pivot only take part for the calculations, corresponding to the remaining elements the transformations by the application of the formula Post.

· To each transformation a Positive sign corresponds to him or Negative according to it is the sign of the cofactor of the Pivot.

· All number that is multiplied by the Pivot will consider Main Diagonal him

Example. To calculate the following Determinant of order 2.

$$\Delta_2 = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}$$

Anyone of the four elements can be "Pivot" we will treat whenever the Pivot is the first element (3) like "Pivot"

$$\Delta_2 = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}^{\div 1} = \frac{3\times 5 - 2\times 4}{1} = 7$$

Example. To calculate the following Determinant of order 3

$$\Delta_3 = \begin{vmatrix} 3 & -2 & 2 \\ 1 & 4 & 5 \\ 6 & -1 & 2 \end{vmatrix}$$

Solution.

$$\Delta_3 = \begin{vmatrix} 3 & -2 & 2 \\ 1 & 4 & 5 \\ 6 & -1 & 2 \end{vmatrix}^{\div 1} \sim \begin{vmatrix} 14 & 13 \\ 9 & -6 \end{vmatrix}^{\div 3} = \frac{(14)(-6)-(9)(13)}{3} = -67$$

Explaining the 4 initial operations of it formulates Post that are:

$$N.E. = \frac{(3)(4)-(1)(-2)}{1} = 14$$

$$N.E. = \frac{(3)(-1)-(6)(-2)}{1} = 9$$

$$N.E. = \frac{(3)(5)-(1)(2)}{1} = 13$$

$$N.E. = \frac{(3)(2)-(6)(2)}{1} = -6$$

NOTE. - If when dividing between the Previous Pivot in the Method Post, it appears a fractional amount, this will be a signal that an error was committed, this is one of the advantages that using has the Method Post (the divisions between the Previous Pivot always are exact).

Example. To calculate the following Determinant of order 4

$$\Delta_4 = \begin{vmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & 2 & -3 \\ 3 & -1 & -1 & 2 \\ 2 & 3 & 1 & 4 \end{vmatrix} \sim$$

Solution.

$$\Delta_4 = \begin{vmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & 2 & -3 \\ 3 & -1 & -1 & 2 \\ 2 & 3 & 1 & 4 \end{vmatrix}^{\div 1} \sim \begin{vmatrix} 3 & 5 & -7 \\ -5 & 1 & 1 \\ 4 & 4 & 6 \end{vmatrix}^{\div 2} \sim \begin{vmatrix} 14 & -16 \\ -4 & 23 \end{vmatrix}^{\div 3} = \frac{(14)(23)-(-4)(-16)}{3} = 86$$

Note: The passages of the development are left for the Student.

ALGORITHM POST APPLIED To The SOLUTION OF SYSTEMS

OF LINEAR EQUATIONS

The Algorithm Post also is applied in the solution of Systems of Linear Equations with whole coefficients, reason why we must have well-taken care of which if some equation is given with decimal or fractional amounts first that we must do it is to transform using it the multiplication by a no null constant so that it leaves only whole numbers us in the equations.

When applying it formulates Post to the Matrix of the System of Equations makes transformations of this to pass it to another Equivalent Matrix that has the same solution that the original one, until reducing it to a system that easily can be solved. The rules to apply are the following ones. Given to a System of n Incognito Equations with n.

Rules.

· The Increased Shade is placed

· They take n Pivots (P.) taking one and single one from each row and column of the coefficients of the incognitos. It is recommended to take the elements from the Main Diagonal like Pivots.

· In each transformation the elements of row of the Pivot they happen equal without being modified.

· The elements of the column of the Pivot become "zeros" except the Pivot.

· The "hollows" that are in the new matrix will be filled with the transformation of the corresponding elements of the previous matrix by means of the application of formulates Post.

Let us apply these rules to the solution of the following ones:

Examples.

$$\begin{array}{ccccccc} 3x & + & y & - & 2z & = & 1 \\ 2x & + & 3y & - & z & = & 2 \\ x & - & 2y & + & 2z & = & -10 \end{array}$$

Solution. We will write the Increased Matrix and we applied the rules.

$$\left(\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ 2 & 3 & -1 & 2 \\ 1 & -2 & 2 & -10 \end{array}\right)^{\div 1} \sim \left(\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ 0 & 7 & 1 & 4 \\ 0 & -7 & 8 & -31 \end{array}\right)^{\div 3} \sim \left(\begin{array}{ccc|c} 7 & 0 & -5 & 1 \\ 0 & 7 & 1 & 4 \\ 0 & 0 & 21 & -63 \end{array}\right)^{\div 7} \sim \left(\begin{array}{ccc|c} 21 & 0 & 0 & -42 \\ 0 & 21 & 0 & 21 \\ 0 & 0 & 21 & -63 \end{array}\right)$$

It completes matrix represents the system of linear equations equivalent:

and its solution is $x = \dfrac{-42}{21} = -2$ $y = \dfrac{21}{21} = 1$ and $z = \dfrac{-63}{21} = -3$

NOTE. If the rule of Cramer were applied to solve this system of equations it would be that: $\Delta s = 21$ $\Delta x = -42$ $\Delta y = 21$ and $\Delta z = -63$ that they are the numbers that appear in the equivalent matrix.

EXERCISES

I Get the value of the determinant:

1) $\begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix}$ 2) $\begin{vmatrix} 2x & 3y \\ 1 & 2 \end{vmatrix}$

3) $\begin{vmatrix} 6 & -2 \\ 0 & 3 \end{vmatrix}$ 4) $\begin{vmatrix} 5 & 4 \\ -2 & 0 \end{vmatrix}$

5) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{vmatrix}$ 6) $\begin{vmatrix} 1 & 3 & 5 \\ -1 & 0 & 2 \\ -1 & 2 & -4 \end{vmatrix}$

7) $\begin{vmatrix} 2 & 3 & 6 \\ 6 & 1 & 2 \\ -2 & 5 & 4 \end{vmatrix}$ 8) $\begin{vmatrix} 1 & 2 & -2 \\ 3 & 5 & 7 \\ -3 & -6 & 6 \end{vmatrix}$

9) $\begin{vmatrix} 1 & 7 & 8 \\ 0 & 5 & -6 \\ 0 & 0 & 9 \end{vmatrix}$

II Solve the following systems using Cramer's rule.

1) $a)\ \ x - 2y + 3z = 1$

$b)\ \ 3x + y - 4z = -9$

$c)\ \ 2x + 5y - z = 6$

2) $a)\ \ 3x + y - 2z = -1$

$b)\ \ x + y + z = 4$

$c)\ \ 4x - 3y + 5z = 26$

3) $a)\ \ 2x - 3y + z = -3$

4) $a)\ \ x + y - z + w = 4$

$b)\ \ 3x + 4y - 2z = 10$ $\qquad$ $b)\ \ 2x - y + z - 3w = 0$

$c)\ \ x + 2y + 4z = 2$ $\qquad$ $c)\ \ x + 2y - z - w = 1$

$d)\ \ x - 3y + z + 2w = 5$

III Compute the inverse of the given matrix.

1) $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$ $\qquad$ 2) $\begin{pmatrix} 2 & -1 & 2 \\ 3 & 5 & -2 \\ 1 & 4 & -1 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & -3 \\ 3 & -2 & -1 \end{pmatrix}$ $\qquad$ 4) $\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 3 \\ 2 & 5 & -2 \end{pmatrix}$

IV Solve systems of equations by applying the inverse of the matrix.

1) $a)\ \ x - 2y + 3z = 1$ $\qquad$ 2) $a)\ \ 3x + y - 2z = -1$

$b)\ \ 3x + y - 4z = -9$ $\qquad$ $b)\ \ x + y + z = 4$

$c)\ \ 2x + 5y - z = 6$ $\qquad$ $c)\ \ 4x - 3y + 5z = 26$

3) $a)\ \ 2x - 3y + z = -3$

$b)\ \ 3x + 4y - 2z = 10$

$c)\ \ x + 2y + 4z = 2$

V Get the value of determinant by using MONTANTE algorithm upright.

1) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{vmatrix}$

2) $\begin{vmatrix} 1 & 3 & 5 \\ -1 & 0 & 2 \\ -1 & 2 & -4 \end{vmatrix}$

3) $\begin{vmatrix} 2 & 3 & 6 \\ 6 & 1 & 2 \\ -2 & 5 & 4 \end{vmatrix}$

4) $\begin{vmatrix} 1 & 2 & -2 \\ 3 & 5 & 7 \\ -3 & -6 & 6 \end{vmatrix}$

5) $\begin{vmatrix} 1 & 7 & 8 \\ 0 & 5 & -6 \\ 0 & 0 & 9 \end{vmatrix}$

VI Solve systems of equations given using the augmented matrix MONTANTE algorithm upright.

1)
$a)\ x - 2y + 3z = 1$
$b)\ 3x + y - 4z = -9$
$c)\ 2x + 5y - z = 6$

2)
$a)\ 3x + y - 2z = -1$
$b)\ x + y + z = 4$
$c)\ 4x - 3y + 5z = 26$

3)
$a)\ 2x - 3y + z = -3$
$b)\ 3x + 4y - 2z = 10$
$c)\ x + 2y + 4z = 2$

4)
$a)\ x + y - z + w = 4$
$b)\ 2x - y + z - 3w = 0$
$c)\ x + 2y - z - w = 1$
$d)\ x - 3y + z + 2w = 5$

ANSWERS

1) -5 2) $4x-3y$ 3) 18 4) 8 5) 74

6) -32 7) 96 8) 0 9) 45

II

1) $x=-1, \quad y=2, \quad z=2$

2) $x=2, \quad y=-1, \quad z=3$

3) $x=1, \quad y=\frac{3}{2}, \quad z=-\frac{1}{2}$

4) $x=2, \quad y=-1, \quad z=-2, \quad w=1$

III

1) $\frac{1}{5}\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$

2) $\frac{1}{21}\begin{pmatrix} 3 & 7 & -8 \\ 1 & -4 & 10 \\ 7 & -9 & 13 \end{pmatrix}$

3) $-\frac{1}{56}\begin{pmatrix} -7 & -5 & -13 \\ -7 & -13 & 11 \\ -7 & 11 & -5 \end{pmatrix}$

4) $Dont\ Havent.$

IV

1) $x=-1, \quad y=2, \quad z=2$

2) $x=2, \quad y=-1, \quad z=3$

3) $x=1, \quad y=\frac{3}{2}, \quad z=-\frac{1}{2}$

V

1) 74 2) -32 3) 96 4) 0 5) 45

VI

1) $x=-1, \quad y=2, \quad z=2$

2) $x=2, \quad y=-1, \quad z=3$

3) $x=1, \quad y=\frac{3}{2}, \quad z=-\frac{1}{2}$

4) $x=2, \quad y=-1, \quad z=-2, \quad w=1$

<u>matrices I</u>

INTRODUCTION.

The utility of the Matrices and its true one to be able in the applications, come from the properties of the same matrices, in the characteristics of a matrix that this serving to us as model to mark some situation on a map in the Physics, Engineering, the Communications, Theory of the Probabilities and in the same Mathematics. Very little] can be declared of its importance in the applications with single handling the Algebra of Matrices can Be denoted of two ways with Hooks [or with Great Round Parentheses (in this I capitulate we will use them of the two ways.

DEFINITION OF A MATRIX.

A matrix is a rectangular mathematical adjustment of elements that can be numbers or letters formed by rows (m) and columns (n) that issues the order to us of the matrix and which imagines generally with capital letters with numerical subscripts ($A_{m\times n}$

With the following representation we will deal to clear the doubts that could have arisen

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13\ldots} & a_{1j\ldots} & a_{1n} \\ a_{21} & a_{22} & a_{23\ldots} & a_{2j\ldots} & a_{2n} \\ a_{i1} & a_{i2} & a_{i3\ldots} & a_{ij\ldots} & a_{in} \\ a_{m1} & a_{m2} & a_{m3\ldots} & a_{mj\ldots} & a_{mn} \end{bmatrix} \text{Matrix of order m x n}$$

in any Matrix the "Main Diagonal" can be distinguished that is the formed one by the elements a_{11} , a_{22} , a_{33} , a_{44}.... a_{mn}

When a Matrix has the same one I number of Rows that the one of Columns forms what it is known like a square matrix which him ú can be represented by a n mere real $(|A|)$ this adjustment will be called Determinant to him.

Examples:

$$A_{2\times 2} = \begin{vmatrix} -5 & 4 \\ -2 & 6 \end{vmatrix} \text{its determinant is } |A| = -22$$

$$B_{3\times 3} = \begin{vmatrix} 3 & 1 & 4 \\ 1 & -2 & -1 \\ 2 & 3 & -5 \end{vmatrix} \text{its determinant is } |A| = 70$$

A great amount of definitions of Matrices exists that must be named, some of them occur next:

TRANSPOSED MATRIX (A^T)

The transposed Matrix is that that respectively obtains when interchanging the Rows by Columns and the Columns by Rows of a Matrix (A) given.

If the order of the Matrix **A** given, it is **then m x** n Transposed his A^T it will be of order **n x m.**

Examples:

$$A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \\ 4 & 3 \end{bmatrix} \text{Transposed his it will be } A^T = \begin{bmatrix} 2 & 1 & 4 \\ 5 & -2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 11 & 20 & 2 & 3 \end{bmatrix} \text{Transposed his it will be } B^T = \begin{bmatrix} 11 \\ 20 \\ 2 \\ 3 \end{bmatrix}$$

FIRST IDENTITY (I_n)

The First Identity is that square matrix that has in its main diagonal elements that are the unit () and the other elements are zeros (0)

Examples, of matrices identity of Order 1, Order 2 and of Order 3 respectively

$$I_1 = \begin{bmatrix} 1 \end{bmatrix} I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{etc.}$$

FIRST STEP

The First Step is the rectangular matrix that has elements "one" in the Main Diagonal and below these Diagonal elements "zero" and by above of this Diagonal any value.

Example:

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & 1 & 9 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{First Step of 3 xs 4}$$

NULL MATRIX (0)

The Null Matrix m x n symbolized by a zero, has m rows and n columns in which all their elements are "zeros" and the flame Identity Additive in where $A+0=A$

$$0_{3\times 2}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

INVERSE MATRIX $\left(A^{-1}\right)$

It is that square Matrix that is obtained from another similar Matrix (A) of such form that when multiplying them in any order gives a First Identity us.

$$A\bullet A^{-1}=I=A^{-1}\bullet A$$

With Matrices can to make almost all operations fundamental that they are made with the real numbers with exception of the division of Matrices, that does not exist in the Algebra of Matrices, next we will initiate the study of these operations with:

EQUALITY OF TWO MATRICES

Two Matrices of order $m\times n$ they are equal if and single if each element of one of them is equal to corresponding element of the other

If and single if $aij=bij$

An example we will be able to clear the existing doubts:

$$\begin{bmatrix}-1 & 7 & \frac{100}{2}\\ 9 & 81 & -10\\ 8 & \sqrt{36} & 4^3\end{bmatrix}=\begin{bmatrix}(-1)^3 & \sqrt{49} & 50\\ 9 & 3^4 & -10\\ \sqrt{64} & 6 & 64\end{bmatrix}$$

The equality of two Matrices can generate **Systems of Linear Equations** as we can see it in the following example:

$$\begin{bmatrix}2x+1 & y-4z\\ y+5x & w-2x\end{bmatrix}=\begin{bmatrix}7 & 6\\ 17 & -2\end{bmatrix}$$

These Matrices are equal yes and single yes

$$x=3 \qquad y=2 \qquad z=-1 \qquad w=4$$

SUM OF TWO MATRICES

In order to be able to make the sum of two Matrices it is necessary that these are of the SAME ORDER and each element of the first Matrix Will be added with the corresponding element of the second matrix we will clarify the previous thing with the following example:

Example: given the Matrices and to B to carry out its sum.

$$A = \begin{bmatrix} 1 & 7 & 3 & 8 \\ 2 & 3 & 4 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 6 & 2 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Solution:

$$A+B = \begin{bmatrix} 1+2 & 7+6 & 3+2 & 8+4 \\ 2+4 & 3+1 & 4+3 & 12+2 \end{bmatrix} = \begin{bmatrix} 3 & 13 & 5 & 12 \\ 6 & 4 & 7 & 14 \end{bmatrix}$$

The operation To reduce a Matrix of another one is similar to the sum of Matrices is only necessary to respect the operations of signs

Example: to calculate the difference (it reduces) of the Matrices To and B.

$$A = \begin{bmatrix} 3 & 4 & 9 & 4 \\ 2 & 6 & 8 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 8 & 3 \end{bmatrix}$$

Solution:

$$A+B = \begin{bmatrix} 3-1 & 4-2 & 9-3 & 4-0 \\ 2-4 & 6-5 & 8-8 & 1-3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -2 & 1 & 0 & -2 \end{bmatrix}$$

Note: The Sum of Matrices is Commutative and also Associative it is to say:

$$A+B = B+A \qquad A+(B+C) = (A+B)+C$$

Multiplication of a matrix by a non-zero constant

A Matrix of order m x n can be multiplied by a number different from zero giving like result another matrix of the same order, being this expression expressed by the following definition:

The product of a Matrix **To** of order m x n by a constant no null **k** is the Matrix **kA** of order m x n that obtains when multiplying each element of A by the constant k giving like result:

$$kA = ka_{ij}$$

Example: if $k = 3$ and $A = \begin{bmatrix} -2 & 3 \\ 4 & 7 \end{bmatrix}$ to calculate kA

$$kA = 3\begin{bmatrix} -2 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 9 \\ 12 & 21 \end{bmatrix}$$

Division of a matrix by a non-NULL constant

A Matrix of order m x n can Be divided by a number different from zero giving like result another matrix of the same order, being this expression expressed by the following definition:

The Quotient of a Matrix **To** of order m x n by a constant no null **k** is the Matrix $\frac{A}{k}$ of order m x n that obtains when dividing each element of A by the constant k giving like result:

$$\frac{A}{k} = \frac{a_{ij}}{k}$$

Example: if $k = 3$ and $A = \begin{bmatrix} 2 & -36 \\ 3 & 45 \end{bmatrix}$ to calculate $\frac{A}{k}$

$$\frac{A}{k} = \frac{\begin{bmatrix} 2 & -36 \\ 3 & 45 \end{bmatrix}}{3} = \begin{bmatrix} \frac{2}{3} & -12 \\ 1 & 15 \end{bmatrix}$$

Multiplication of two MATRICES.

In order to multiply two matrices A and B he is **REQUISITE** (necessary) that the number of columns of the first matrix is equal to the number of rows of the second matrix, obtaining a resulting Matrix that will be formed with the number of rows of the first Matrix (m) and with the number of columns of the second Matrix (p). If C is the product of A • Then B:

$$\underset{m\times n}{A} \bullet \underset{n\times p}{B} = \underset{m\times p}{C} \quad \text{Requirement } n = n$$

Example. To calculate the product of matrices A by B of being possible.

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 5 & 4 & 1 & 6 \\ 0 & -3 & -2 & 5 \end{bmatrix}$$

2 xs **3** **3** xs 4

Observe that the columns of A and the lines of B are equal; therefore, yes the multiplication can be carried out being the resulting matrix (c) with 2 lines and 4 columns. The calculations of the elements of C are the following ones.

C_{11} = (3)(1)+(-2)(5)+(5)(0) = -7

C_{12}= (3)(2)+(-2)(4)+(5)(-3)=-17

C_{13}= (3)(-3)+(-2)(1)+(5)(-2)=-21

C_{14}= (3)(0)+(-2)(6)+(5)(5)=13

C_{21}= (1)(1)+(0)(5)+(4)(0)=1

C_{22}= (1)(2)+(0)(4)+(4)(-3)=-10

C_{23}= (1)(-3)+(0)(1)+(4)(-2)=-11

C_{24}= (1)(0)+(0)(6)+(4)(5)=20

$$A \bullet B = C = \begin{bmatrix} -7 & -17 & -21 & 13 \\ 1 & -10 & -11 & 20 \end{bmatrix} \text{Result}$$

It will be possible to be observed at first that the product of two matrices is not commutative. In effect in the multiplication previous To x B if it were possible to be made but if we tried to multiply B x To, we will realize that is not possible to make it.

The associative law is fulfilled as long as the multiplication of the matrices is defined. This implies that the number of columns of A is equal to the number of lines of B, in addition the number of columns of B will have to be equal to the number of lines of C and the resulting matrix will have the number of lines of A and the number of columns of C, symbolically is represented as it follows:

$$A \times (B \times C) = (A \times B) \times C = D$$

mxn nxp pxq = mxn nxp pxq = mxq

The multiplication of matrices also fulfills the law Distributive as long as the involved matrices have the suitable number of lines and columns as in the previous case being represented as it follows:

$$To \times (B + C) = To \times B + To \times C$$

This is fulfilled if To it is of order **mxn** and B and C of **nxp**

$$(A+B) \times C = To \times C + B \times C$$

That one is fulfilled if To and B is of **mxn** and C of **nxp** .

When all the matrices are square the properties associative and distributive they verify, this considers as a special case of the multiplication of matrices.

SOLUTION OF SYSTEMS OF FIRST LINEAR EQUATIONS USING.

The systems of linear equations can be solved by the method of Elimination by Sum and Subtraction the one of Substitution the one of Equalization or the one of Graphs . In this subject an alternative will occur him to the student one more than she is the one to use the matrices for his solution.

Let us suppose that one occurs to a system of n incognito equations us with n.

$$\begin{array}{ccccccccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & + & . & . & . & + & a_{1n}w & = & k_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & + & . & . & . & + & a_{2n}w & = & k_2 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ a_{n1}x & + & a_{n2}y & + & a_{n3}z & + & . & . & . & + & a_{nn}w & = & k_n \end{array}$$

This system can be represented by a matrix as it follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} & k_1 \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2n} & k_2 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & a_{nn} & k_n \end{bmatrix}$$

This matrix that represents the system of linear equations with the ordered equations is called the INCREASED MATRIX , also we can distinguish other two matrices in the same system that are:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & a_{nn} \end{bmatrix} \text{MATRIX OF THE COEFFICIENTS}$$

And

$$\begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix}$$ MATRIX OF THE CONSTANT TERMS

ELEMENTARY TRANSFORMATIONS OF LINE.

In systems of linear equations we can conduct the following operations.

1. To interchange an equation by another one

2. To multiply an equation by a no null constant

3. To add two equations (or to reduce)

4. To multiply an equation by a constant and the product to add it

to another one of the equations.

These operations conducted with the equations are with the objective to form " Systems Equivalent " to the system since it has the same solution that the original system and whose solution is but easy to obtain.

The same operations conducted to the "Equations" can be made in the matrices on the "Lines" being known with the name of elementary transformations of line of a matrix and are the following ones.

ELEMENTARY TRANSFORMATIONS OF LINE

1. Interchange of two lines

2. Multiplication of all the elements of a line by a constant different from zero

3. Multiplication of a line by a no null constant and the product to add it to the corresponding element of any other line.

GAUSSIAN ELIMINATION

The process consists of forming a "Stepped Matrix" that has elements "Zero" below the main diagonal and elements "or Zeros" in the main diagonal (of preference) that will represent the system of "Equivalent Equations" with which it will be possible to be solved "in reverse" or of the last equation very easily until the first system of equations.

Next the solution of a system of 3 will be obtained linear equations of x3 in which it will express with words the conducted operations

Example. To solve the System of Linear Equations given.

$$\begin{aligned} 2x + y - 4z &= 3 \\ x - 2y + 3z &= 4 \\ -3x + 4y - z &= -2 \end{aligned}$$

Solution. The increased matrix is written

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & 3 \\ 1 & -2 & 3 & 4 \\ -3 & 4 & -1 & -2 \end{array}\right]$$

Rows 1 and 2 interchange

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & 1 & -4 & 3 \\ -3 & 4 & -1 & -2 \end{array}\right]$$

Now each element of Line 1 by (-2) and the Extreme Result with the corresponding element of Line 2 Is multiplied, and also To multiply each element of Line 1 by (3) and the turn out To add it with the corresponding element of Line 3 and we has left:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 5 & -10 & -5 \\ 0 & -2 & 8 & 10 \end{array}\right]$$

It is multiplied by (1/5) Line 2 and we have left:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & -2 & 8 & 10 \end{array}\right]$$

Line 2 by (2) Is multiplied and Extreme to Line 3 and we have left ourselves:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Finally line 3 is multiplied by (1/4) and we obtain:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This last matrix represents the Equivalent System of Equations:

placing the Variables to the Coefficients we have left:

$$\begin{array}{ccccccc} x & - & 2y & + & 3z & = & 4 \\ & & y & - & 2z & = & -1 \\ & & & & z & = & 2 \end{array}$$

And solving in "reverse" this system they looked for solution is obtained.

$z = 2$, $y = 3$ and $x = 4$.

As it is already known a system of equations can have: **Unique solution** (like the one of the previous example), **Not to have Solution** or **To have an Infinite Number of Solutions.**

SYSTEMS OF LINEAR EQUATIONS HOMOGENOUS, DEFECTIVE And REDUNDANTE.

Systems of Linear Equations exist that can be considered like special, these are the Homogenous, Defective linear systems and the Redundant, which we will treat next.

A system of equations is **HOMOGENOUS** if all their equations are even to zero, is to say the constant terms are all zero.

A Homogenous system of equations has like solution which is known with the name of **TRIVIAL SOLUTION** that is obtained giving the value of zero to all the variables, as long as the Delta of the System is Different from Zero.

But in some cases the mentioned Systems have solutions non-trivial and have an Infinite Number of Solutions. The form to know if this System of Equations has Solutions non-trivial Is calculating the Delta of the System (Δ_s), if this Delta is Equal to Zero, two options exist

· It does not have Solution and he is Undetermined

· It has an Infinite Number of Solutions

Next the solution of one of them acquires knowledge.

Example. To determine the solution of the given Homogenous system.

$$\begin{array}{rcrcrcl} x & - & y & + & 4z & = & 0 \\ 2x & + & y & - & z & = & 0 \\ -x & - & y & + & 2z & = & 0 \end{array}$$

Solution. - As it were already mentioned the system has like solution $x = 0$, $y = 0$ and $z = 0$ (trivial solution) if the Delta of the System is Different from Zero.

But if they exist solutions non-trivial will be come as it follows: the increased matrix is written and the elementary operations of Line are made.

$$\left[\begin{array}{rrr|r} 1 & -1 & 4 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right]$$

Line 1 Is multiplied by (-2) and later Extreme with Row 2, also Line 1 Is multiplied by (1) and later Extreme with Line 3 and we obtain ourselves:

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & -2 & 6 & 0 \end{array}\right]$$

now Line 2 Is multiplied by (1/3),

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 6 & 0 \end{array}\right]$$

now Line 2 Is multiplied by (2), and the Result add to Line 3

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Matrix that represents the equivalent system of equations:

$$\begin{aligned} &x - y + 4z = 0 \\ &y - 3z = 0 \end{aligned}$$

from equation 2 it is obtained:

$$y = 3z$$

and of equation 1:

$$x = y - 4z$$

Where it is possible to be observed that for any value of **z** one can be obtained of **y** and with **z** and to obtain **x** being left the solution general as it follows:

$$x = y - 4r \qquad y = 3r \qquad z = r$$

where **r** is any real number.

Some particular solutions are

Yes

$r = 1$	$r = 2$	$r = 3$
$z = 1$	$z = 2$	$z = -3$
$y = 3$	$y = 6$	$y = -9$
$x = -1$	$x = -2$	$x = 3$

etc.

A system of equations is **<u>defects</u>** when the number of Incognitos is Greater than number of Equations; as special case treated the one that has **n** Incognito **and** n+1. Equations

All the Defective systems have an infinite number of solutions, since also they can be considered like "Dependent Systems", which as its name says it, the values of some of variables "depend" on the assigned value to one of them who are considered like independent or constant term.

Example. To obtain the general solution when solving the following system defective **Two Solutions Will occur** .

$$2x+3y+4z=1$$
$$3x+4y+5z=3$$

Solution 1.

The Increased matrix is written that represents the system and the necessary transformations of line are made.

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 3 \end{array}\right]$$

now Line 1 Is multiplied by (1/2),

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 3 & 4 & 5 & 3 \end{array}\right]$$

also Line 1 by (-3) Will be multiplied and the Result Will be added with Line 2

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & -\frac{1}{2} & -1 & \frac{3}{2} \end{array}\right]$$

now Equation 2 Is multiplied by (-2).

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 1 & 2 & -3 \end{array}\right]$$

this last matrix represents the following system of equations equivalent:

$$x-z=5$$
$$y+2z=-3$$

of equation 1 and 2 it is possible to be cleared **x** and respectively based on **z** , being the general solution:

$$x = 5 + z \qquad y = -3 - 2z$$

Some of the particular solutions are:

If:

$$\begin{array}{lll} z = 1 & z = 2 & z = -1 \\ y = -5 & y = -7 & y = -1 \\ x = 6 & x = 7 & x = 4 \end{array}$$

etc...

Solution 2.

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ 3x + 4y + 5z &= 3 \end{aligned}$$

it is transferred alongside straight of the Equation one of the Variables and we will consider a Constant

$$\begin{aligned} 2x + 3y &= 1 - 4z \\ 3x + 4y &= 3 - 5z \end{aligned}$$

The Deltas calculate $\Delta_s, \Delta_x y \Delta_y$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1$$

$$\Delta_x = \begin{vmatrix} 1-4z & 3 \\ 3-5z & 4 \end{vmatrix} = 4 - 16z - 9 + 15z = -5 - z$$

$$\Delta_y = \begin{vmatrix} 2 & 1-4z \\ 3 & 3-5z \end{vmatrix} = 6 - 10z - 3 + 12z = 3 + 2z$$

Using the Rule of Cramer:

$$x = \frac{\Delta_x}{\Delta} = \frac{-5 - z}{-1} = 5 + z$$

$$y = \frac{\Delta_y}{\Delta} = \frac{3 + 2z}{-1} = -3 - 2z$$

As it is possible to be observed the Resulting Equations are Equal to which we found previously now if we assigned any Value to him "to z"

$$\begin{array}{lll} z=1 & z=2 & z=-1 \\ x=6 & x=7 & x=4 \\ y=-5 & y=-7 & y=-1 \end{array}$$

A system of equations **REDUNDANT** is characterized to have But Equations that he number of Incognitos. The special case in that will study they exist **n** Incognito Equations **and** n-1. An example of a redundant system of 4 Equations with 3 Incognitos is the following one

$$\begin{aligned} &x+2y-z=2 \\ &2x+5y-3z=3 \\ &4x+9y-6z=4 \\ &-x-4y+2z=-3 \end{aligned}$$

A Condition so that a System of Linear Equations Redundant is Compatible is that the Determinant formed by the Coefficients and the Constants is equal to ZERO

Firstly it will be come to calculate the value of the Determinant and knowledge if it fulfills the Condition, to thus be able later to find the Values of the Variables

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -3 & 3 \\ 4 & 9 & -6 & 4 \\ -1 & -4 & 2 & -3 \end{vmatrix} = 0$$

in the solution of the previous Determinant the Method of Cofactors will be used

$$\begin{aligned} &1\begin{vmatrix} 5 & -3 & 3 \\ 9 & -6 & 4 \\ -4 & 2 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 & 2 \\ 9 & -6 & 4 \\ -4 & 2 & -3 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -4 & 2 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ 9 & -6 & 4 \end{vmatrix} = 0 \\ &= 1(90+48+54-72-40-81) - 2(36+16+36-48-16-27) + \\ &+ 4(18+12+20-24-12-15) + 1(-24-27-60+54+20+36) = 0 \\ &= 1(-1) - 2(-3) + 4(-1) + 1(-1) = 0 \\ &-1+6-4-1=0 \\ &\therefore \\ &0=0 \end{aligned}$$

as if System Fulfills the Condition east is Compatible and it is possible to be solved eliminating one of Equations and solving the normal system of equations, (n equations, with n incognito) for finally, to see if "the eliminated" equation has the found solution.

Exercises

I Get the transposed matrix:

1) $A = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$

3) $C = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 9 & 7 \end{bmatrix}$

2) $B = \begin{bmatrix} 1 & -4 & 7 & 0 \end{bmatrix}$

4) $D = \begin{bmatrix} 3 & 5 & -2 \\ 4 & 7 & 6 \\ 0 & 8 & 1 \end{bmatrix}$

II Obtain the values of x, y, z, and w which make that matrices are equivalent.

$$\begin{bmatrix} 3x-1 & y+2x \\ w+2x & 3y-w \end{bmatrix} = \begin{bmatrix} 11 & 7 \\ 5 & 0 \end{bmatrix}$$

III Given matrices A, B and C get the following operations.

$A = \begin{bmatrix} 5 & 2 \\ -1 & 0 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -4 \\ 3 & 5 \\ 1 & -2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 \\ 8 & 6 \\ 7 & -9 \end{bmatrix}$

1) $A + B$

3) $(A + B) - (A - C)$

2) $A - C$

IV Multiplying the matrix by the constant k.

1) $A = \begin{bmatrix} 5 & -3 \\ 0 & 4 \\ -1 & 6 \end{bmatrix}$ $k = 4$

2) $B = \begin{bmatrix} -1 & 7 & -5 & 3 \\ 8 & -4 & 2 & 0 \end{bmatrix}$ $k = -2$

V Divide the matrix by the constant k.

1) $\begin{bmatrix} 2 & -4 & 8 & 6 \\ 1 & -6 & 3 & -2 \end{bmatrix} \quad k = 4$

2) $\begin{bmatrix} 18 & -3 \\ 0 & 6 \\ -5 & -24 \end{bmatrix} \quad k = -3$

VI Divide the matrix by the constant k.

$A_{2x2} = \begin{bmatrix} 5 & 3 \\ -2 & 4 \end{bmatrix} \quad B_{3x2} = \begin{bmatrix} -1 & 8 \\ 0 & 7 \\ -3 & 2 \end{bmatrix} \quad C_{3x3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$D_{3x3} = \begin{bmatrix} -2 & 7 & 3 \\ 5 & 9 & 10 \\ 6 & -4 & 1 \end{bmatrix} \quad E_{2x3} = \begin{bmatrix} 9 & 2 & -1 \\ 5 & -3 & 10 \end{bmatrix}$

1) $B \bullet A$
2) $A \bullet B$
3) $B \bullet E$
4) $E \bullet B$
5) $C \bullet D$
6) $D \bullet C$
7) $D \bullet B$
8) $B \bullet D$
9) $E \bullet D$
10) $D \bullet E$
11) $E \bullet C$
12) $C \bullet E$

VII Solve the following system using transformations of space and Gaussian elimination.

1)
$a) \quad x + y - z = -2$
$b) \quad 2x - 3y + 2z = 14$
$c) \quad x - 2y + 3z = 12$

2)
$a) \quad 3x - y + 2z = 11$
$b) \quad 2x + 2y - z = -5$
$c) \quad x - 3z = -8$

VIII Solve the following system.

1) $a)\quad x + 2y - 3z = 0$

$b)\quad 5x - 2y - 4z = 0$

$c)\quad x - 6y + 2z = 0$

2) $a)\quad 4x + 3y + z = 2$

$b)\quad 3x - 5y + 2z = 4$

3) $a)\quad x - 3y + 2z = -4$

$b)\quad 2x + y - 4z = 16$

$c)\quad 3x + 5y - 2z = 12$

$d)\quad 4x - 2y + 5z = -7$

IX Get the value of the determinant:

1) $\begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix}$

2) $\begin{vmatrix} 2x & 3y \\ 1 & 2 \end{vmatrix}$

3) $\begin{vmatrix} 6 & -2 \\ 0 & 3 \end{vmatrix}$

4) $\begin{vmatrix} 5 & 4 \\ -2 & 0 \end{vmatrix}$

5) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{vmatrix}$

6) $\begin{vmatrix} 1 & 3 & 5 \\ -1 & 0 & 2 \\ -1 & 2 & -4 \end{vmatrix}$

7) $\begin{vmatrix} 2 & 3 & 6 \\ 6 & 1 & 2 \\ -2 & 5 & 4 \end{vmatrix}$

8) $\begin{vmatrix} 1 & 2 & -2 \\ 3 & 5 & 7 \\ -3 & -6 & 6 \end{vmatrix}$

9) $\begin{vmatrix} 1 & 7 & 8 \\ 0 & 5 & -6 \\ 0 & 0 & 9 \end{vmatrix}$

X Solve the following systems using Cramer's rule.

1)
$a)\ \ x - 2y + 3z = 1$
$b)\ \ 3x + y - 4z = -9$
$c)\ \ 2x + 5y - z = 6$

2)
$a)\ \ 3x + y - 2z = -1$
$b)\ \ x + y + z = 4$
$c)\ \ 4x - 3y + 5z = 26$

3)
$a)\ \ 2x - 3y + z = -3$
$b)\ \ 3x + 4y - 2z = 10$
$c)\ \ x + 2y + 4z = 2$

4)
$a)\ \ x + y - z + w = 4$
$b)\ \ 2x - y + z - 3w = 0$
$c)\ \ x + 2y - z - w = 1$
$d)\ \ x - 3y + z + 2w = 5$

XI Compute the inverse of the given matrix.

1) $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$

2) $\begin{pmatrix} 2 & -1 & 2 \\ 3 & 5 & -2 \\ 1 & 4 & -1 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & -3 \\ 3 & -2 & -1 \end{pmatrix}$

4) $\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 3 \\ 2 & 5 & -2 \end{pmatrix}$

XII Solve systems of equations by applying the inverse of the matrix.

1) $a)\ \ x - 2y + 3z = 1$

$b)\ \ 3x + y - 4z = -9$

$c)\ \ 2x + 5y - z = 6$

2) $a)\ \ 3x + y - 2z = -1$

$b)\ \ x + y + z = 4$

$c)\ \ 4x - 3y + 5z = 26$

3) $a)\ \ 2x - 3y + z = -3$

$b)\ \ 3x + 4y - 2z = 10$

$c)\ \ x + 2y + 4z = 2$

XIII Get the value of determinant by using algorithm upright.

1) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{vmatrix}$

2) $\begin{vmatrix} 1 & 3 & 5 \\ -1 & 0 & 2 \\ -1 & 2 & -4 \end{vmatrix}$

3) $\begin{vmatrix} 2 & 3 & 6 \\ 6 & 1 & 2 \\ -2 & 5 & 4 \end{vmatrix}$

4) $\begin{vmatrix} 1 & 2 & -2 \\ 3 & 5 & 7 \\ -3 & -6 & 6 \end{vmatrix}$

5) $\begin{vmatrix} 1 & 7 & 8 \\ 0 & 5 & -6 \\ 0 & 0 & 9 \end{vmatrix}$

XIV Solve systems of equations given using the augmented matrix algorithm upright.

1) $a)\ \ x - 2y + 3z = 1$

2) $a)\ \ 3x + y - 2z = -1$

$b)\ \ 3x + y - 4z = -9$

$c)\ \ 2x + 5y - z = 6$

$b)\ \ x + y + z = 4$

$c)\ \ 4x - 3y + 5z = 26$

3) $a)\ \ 2x - 3y + z = -3$

$b)\ \ 3x + 4y - 2z = 10$

$c)\ \ x + 2y + 4z = 2$

4) $a)\ \ x + y - z + w = 4$

$b)\ \ 2x - y + z - 3w = 0$

$c)\ \ x + 2y - z - w = 1$

$d)\ \ x - 3y + z + 2w = 5$

SOLUTIONS

I

1) $A^T = \begin{bmatrix} 3 & -2 & 4 \end{bmatrix}$

3) $C^T = \begin{bmatrix} 2 & 3 \\ -1 & 9 \\ 5 & 7 \end{bmatrix}$

2) $B^T = \begin{bmatrix} 1 \\ -4 \\ 7 \\ 0 \end{bmatrix}$

4) $D^T = \begin{bmatrix} 3 & 4 & 0 \\ 5 & 7 & 8 \\ -2 & 6 & 1 \end{bmatrix}$

II

$x = 4,\ \ y = -1,\ \ z = 2,\ \ w = -3$

III

1) $\begin{bmatrix} 11 & -2 \\ 2 & 5 \\ 5 & -5 \end{bmatrix}$

3) $\begin{bmatrix} 8 & -4 \\ 11 & 11 \\ 8 & -11 \end{bmatrix}$

2) $\begin{bmatrix} 3 & 2 \\ -9 & -6 \\ -3 & 6 \end{bmatrix}$

IV

1) $\begin{bmatrix} 20 & -12 \\ 0 & 16 \\ -4 & 24 \end{bmatrix}$

2) $\begin{bmatrix} 2 & -14 & 10 & -6 \\ -16 & 8 & -4 & 0 \end{bmatrix}$

V

1) $\begin{bmatrix} \frac{1}{2} & -1 & 2 & \frac{3}{2} \\ \frac{1}{4} & -\frac{3}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$

2) $\begin{bmatrix} -6 & 1 \\ 0 & -2 \\ \frac{5}{3} & 8 \end{bmatrix}$

VI

1) $\begin{bmatrix} -21 & 29 \\ -14 & 28 \\ -19 & -1 \end{bmatrix}$

2) No es posible

3) $\begin{bmatrix} 31 & -26 & 81 \\ 35 & -21 & 70 \\ -17 & -12 & 23 \end{bmatrix}$

4) $\begin{bmatrix} -6 & 84 \\ -35 & 39 \end{bmatrix}$

5) $\begin{bmatrix} -2 & 7 & 3 \\ 5 & 9 & 10 \\ 6 & -4 & 1 \end{bmatrix}$

6) $\begin{bmatrix} -2 & 7 & 3 \\ 5 & 9 & 10 \\ 6 & -4 & 1 \end{bmatrix}$

7) $\begin{bmatrix} -7 & -39 \\ -35 & 123 \\ -9 & 22 \end{bmatrix}$

8) No es posible

9) $\begin{bmatrix} -14 & 85 & 46 \\ 35 & -32 & -5 \end{bmatrix}$

10) No es posible

11) $\begin{bmatrix} 9 & 2 & -1 \\ 5 & -3 & 10 \end{bmatrix}$

12) No es posible

VII

1) $x = 2,\ y = -2,\ z = 2$

2) $x = 1,\ y = -2,\ z = 3$

VIII

1) $x = 2,\ y = 1,\ z = 2$

3) $x = 2,\ y = 0,\ z = -3$

2) $x = \frac{-22+11z}{-29}, \quad y = \frac{10-5z}{-29}$

XI

1) -5

2) $4x - 3y$

3) 18

4) 8

5) 74

6) -32

7) 96

8) 0

9) 45

X

1) $x = -1, \quad y = 2, \quad z = 2$

2) $x = 2, \quad y = -1, \quad z = 3$

3) $x = 1, \quad y = \frac{3}{2}, \quad z = -\frac{1}{2}$

4) $x = 2, \quad y = -1, \quad z = -2, \quad w = 1$

XI

1) $\frac{1}{5}\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$

2) $\frac{1}{21}\begin{pmatrix} 3 & 7 & -8 \\ 1 & -4 & 10 \\ 7 & -9 & 13 \end{pmatrix}$

3) $-\frac{1}{56}\begin{pmatrix} -7 & -5 & -13 \\ -7 & -13 & 11 \\ -7 & 11 & -5 \end{pmatrix}$

4) $Don\acute{}t\ Have.$

XII

1) $x = -1, \quad y = 2, \quad z = 2$

2) $x = 2, \quad y = -1, \quad z = 3$

3) $x = 1, \quad y = \frac{3}{2}, \quad z = -\frac{1}{2}$

XIII

1) 74 2) -32 3) 96 4) 0 5) 45

XIV

1) $x = -1, \quad y = 2, \quad z = 2$

2) $x = 2, \quad y = -1, \quad z = 3$

3) $x = 1, \quad y = \frac{3}{2}, \quad z = -\frac{1}{2}$

4) $x = 2, \quad y = -1, \quad z = -2, \quad w = 1$

Vectors 1

Amount To scale and Vectorial Amount

Amount To climb. It is a magnitude that is specified by a number (and its respective unit), all the numbers you will climb are Real, and they imagine by a letter that can be (capital or very small) **to** or **for** an example of an amount climbing it could be, Mass, Volume, Energy, Work etc.

Vectorial Amount. In order to be able to specify it is needed to know aside from his Magnitude the Direction and its Sense. One imagines by a letter made a will, or made a will

with an arrow $\bar{a}$ or $\vec{A}$ or $\vec{a}$ (some text books indicate with very small letters in black **a** as an example we can mention Force, Speed, Displacement, Acceleration to it, etc.

The Vectors can also be defined as ordered pair of numbers.

The Length of a Vector is what measures the Magnitude of the Vector and imagines of the following way.

$\vec{PQ}$ or $\overline{PQ}$ or $|\vec{PQ}|$

in where to this module also completes is called Norm to him (of $\vec{a}$)

Point of Beginning Full stop

Types of Vectors

Own Vectors: They are those that have Magnitude, Direction and Sense.

Sliding Vectors: They are those that can be moved.

Anchored vectors (fixed): They are those that cannot be moved and has the same tail or point of application.

Equal Vectors: They are those that have the same Magnitude Direction and Sense.

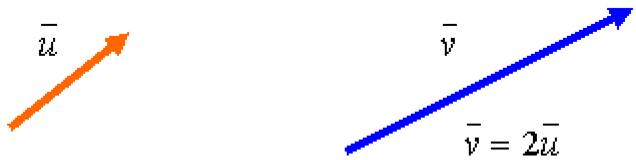
Unitary Vector: He is that that has the equal module to the unit

($\|\vec{u}\| = 1$)

Null Vectors: They are those that measure ZERO and they do not have Magnitude.

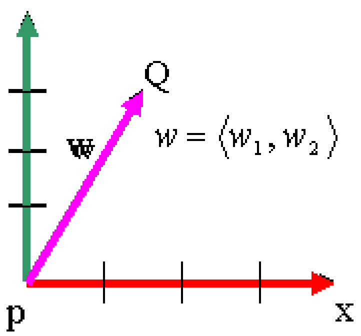
Parallel Vectors: Two no null Vectors $\bar{u}$ and $\bar{v}$ they are PARALLEL if Scalars c exists so that: $\bar{u} = c\bar{v}$

Example. If we have 2 Vectors



Definition and Representation of a Vector in the Plane

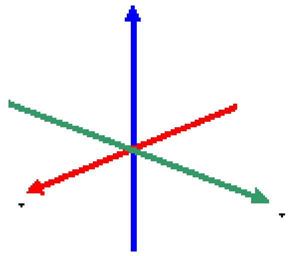
If w is a vector in a plane with starting point in the origin and full stop (w_1 w_2) in where w_1 and w_2 they are the components of w



Definition and Representation of a Vector in the Space

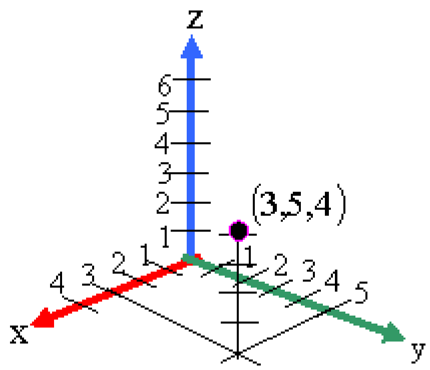
Until today we have exclusively handled systems of coordinates in two dimensions, reason why now one will introduce systems of coordinates in three dimensions.

In order to extend the concept of two to three dimensions we will introduce a three-dimensional system of coordinates, placing a perpendicular **z-axis** in the origin to x-axis and as one is in the following figure.

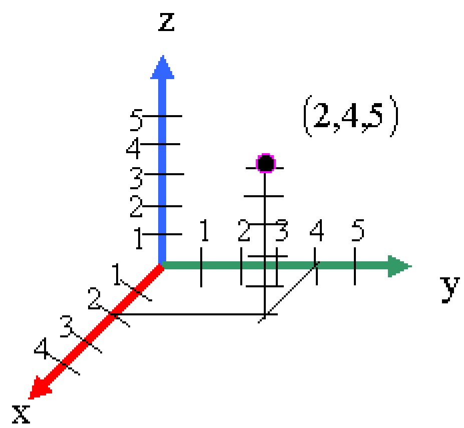


In order to be able to represent a point in the space it is required of a ordered Trio of Numbers that contains a value for X as much, for and like for Z

Example 1: Draw up the following point $(3,5,4)$.

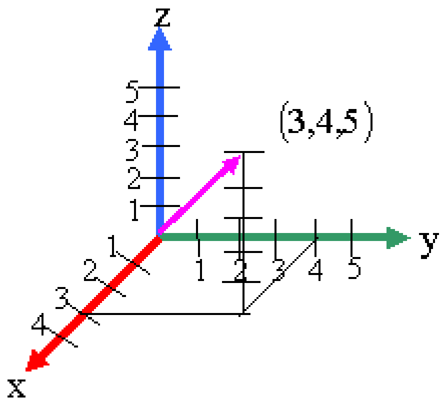


Example 2: Draw up the following point (2, 4, 5) to explain this example we will turn the axes to try to see a little clearer the positioning of the point

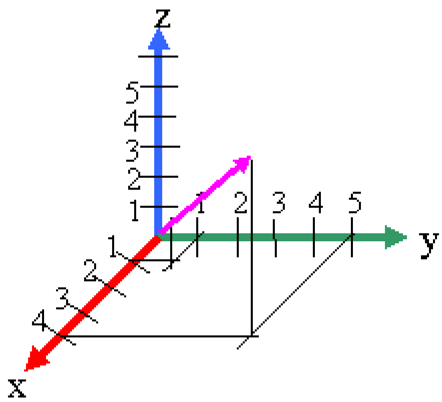


In order to be able to represent a Vector in the Space it is required of Two ordered Trios of Numbers one of which it full stop represents the Point of Beginning of the Vector and the other of the Vector.

Example 1: To represent the following Vector that has like coordinates in the Point of Beginning the Origin (0, 0, 0) and in Full stop (3, 4, 5)



Example 2: To represent the following Vector that has like coordinates the Point of Beginning (1, 1, 1) and in Full stop (4, 5, 3)



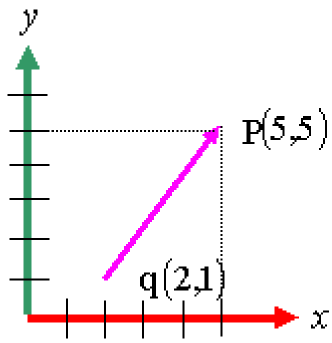
Length of a Vector in the Plane and the Space

Length of a Vector in the Plane:

The length of a Vector as it said previously is what measures the Magnitude and it is possible to be calculated of the following way

$$|pq\| = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$$

Example: To calculate the Length of the following Vector that has like coordinates the P point (5,5) and point q (2,1)



$$|pq\| = \sqrt{(2-5)^2 + (1-5)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

Length of a Vector in the Space:

The length of a Vector as it said previously is what measures the Magnitude and many of Formulas that is been worth for the Plane (two dimensions) also are it for the Space (three dimensions), as for example it is enough to apply the Theorem of Pythagoras twice and with that it is obtained formulates it to find the length between two points in the Space.

$$Longitud = \|d\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example: to calculate the Length between the Points (1, 2, 1) and (-1, 3, -3)

$$\|d| = \sqrt{(-1-1)^2 + (3-2)^2 + (-3-1)^2} = \sqrt{4+1+16} = \sqrt{21}$$

Direction of a Vector

It measures the Inclination of the Segment (the Vector)

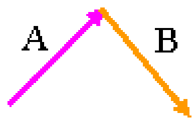
Vectors 11

Sum and Subtraction of Vectors (Graphically)

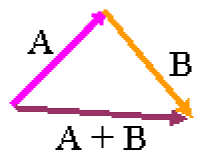
The Sum of two Vectors (Graphically) it is made sliding one of the Vectors and making agree the Point of Beginning of one with Full stop of the other and the Sum it will be the Vector that forms when uniting the Point of Beginning of one with Full stop of the other

Example: To carry out the sum of the following Vectors

If we slid to Vector B and the Point of Beginning we make it agree with him Full stop of the Vector would have left:



∴ The serious Sum the Vector that forms when uniting the Point of Beginning of the Vector To with Full stop of Vector B



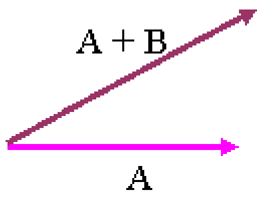
Note: in the following point one demonstrated that it is the same to add

A + B that B + A

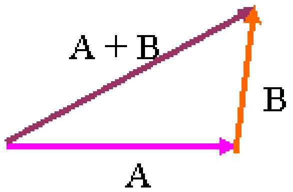
Rested of two Vectors (Graphically) is made making both agree the Point of Beginning of Vectors that are going away To reduce and Rested will be the Vector that forms when uniting the Final Points.

Example: To carry out Rested of the following Vectors.

Solution: if we slid the Vector To and we make agree the Points of Beginning of each Vector we would have left:

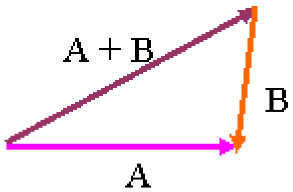


Serious **Subtraction** the Vector that both forms when uniting the Final Points of Vectors



B = (A + B) - A

or



B = A - (A + B)

Note: one is due to make notice that the sense of vector B changes depending like I know this carrying out Rested. (the sense of Vector B always is towards the Vector that is Positive)

Sum and Subtraction of Vectors in a Plane and the Space (Analytically)

In the Plane

I already explain myself previously as one takes place the Sum or Subtraction of two Vectors of Graphical way therefore now it will be explained of the Analytical way.

If we have two vectors To and B whose components are respectively $A = \langle x_1, y_1 \rangle$ and $B = \langle x_2, y_2 \rangle$

Its serious Sum:

$$A + B = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$$

and its serious **Subtraction**:

$$A - B = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle = \langle x_1 - x_2, y_1 - y_2 \rangle$$

which we can understand it of the following way:

The components of X either of and directly Add or Reduce one with the other.

In the Space

Of equal way that in the plane the Sum and Subtraction of two Vectors in the Space To and B whose components are respectively $A = \langle x_1, y_1, z_1 \rangle$ and $B = \langle x_2, y_2, z_2 \rangle$

Its serious Sum:

$$A + B = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

and its serious **Subtraction**:

$$A - B = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$

With some examples the doubts were clarified

Given the Vectors $A = \langle \mathbf{3{,}4} \rangle$ and $B = \langle \mathbf{-2{,}5} \rangle$ to find:

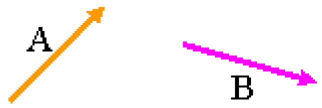
$$A + B = \langle \mathbf{1{,}9} \rangle \text{ and } A - B = \langle \mathbf{5,-1} \rangle \text{ and } B - A = \langle \mathbf{-5{,}1} \rangle$$

Given the Vectors $A = \langle 2,-3,4 \rangle$ and $B = \langle 3,5,-5 \rangle$ to find:

$$A + B = \langle \mathbf{5,2,-1} \rangle \text{ and } A - B = \langle \mathbf{-1,-8,9} \rangle \text{ and } B - A = \langle \mathbf{-1,8,-9} \rangle$$

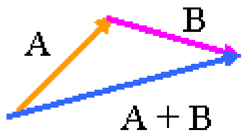
Properties that must fulfill the Sum and Subtraction

If we have two Vectors



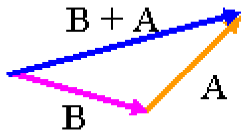
And one requests us A + B

Sliding the Point of Beginning of B to Full stop of A and uniting the Point of Beginning of ***A*** with Full stop of **B** we would have left:



On the other hand if it requests B to us + A

Sliding the Point of Beginning of A to Full stop of B and uniting the Point of Beginning of **B** with Full stop of ***A*** we would have left:

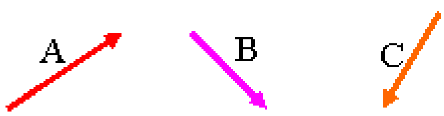


Of which we can see that:

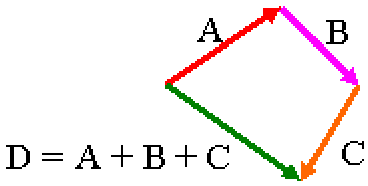
To + B = B + A

Therefore the Sum is Commutative

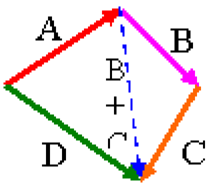
If we have three Vectors



and one requests us A + B + C we would obtain D

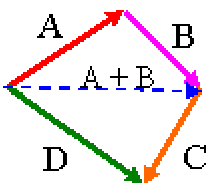


But if they request us To + (B + C), as B + C is the union of the Point of Beginning of B with Full stop of C we would have left D:



D = To + (B + C)

but if + C are requested to us (To + B), as To + B it is the union of the Point of Beginning of A with Full stop of B we would have left D:



D = (To + B) + C

Therefore The Sum Is Associative

Also it is important to make notice that the operations that are made in the Sum, Subtraction and the Multiplication are of the CLOSED **type** .

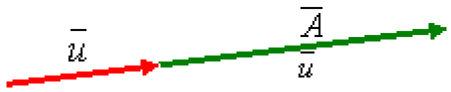
vectors 3

Definition of a Unitary Vector

One says that a Vector is Unitary when its Module (norm) is equal to 1 or to the unit and it is denoted of the following way:

$$\|\bar{u}\| = 1 \text{ (Unitary)}$$

A Unitary Vector has the same Direction and Sense That another Vector that is the Plane or in the Space



In where:

$$\bar{u} = \frac{\bar{A}}{\|\bar{A}\|}$$

Example: Find a Unitary Vector with the same direction of $a = \langle -3,-4 \rangle$

$$\|a\| = \langle -3,-4 \rangle = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

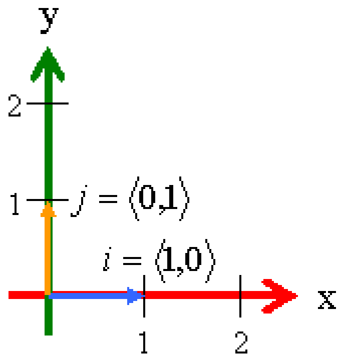
$$\bar{u} = \frac{a}{\|a\|} = \frac{\langle -3,-4 \rangle}{5} = \left\langle \frac{-3}{5}, \frac{-4}{5} \right\rangle$$

Canonical Unitary vectors in a Plane

The Unitary Vectors $\langle 1,0 \rangle$ and $\langle 0,1 \rangle$ Canonical Unitary Vectors are called and they are denoted by:

$$i = \langle 1,0 \rangle \text{ and } j = \langle 0,1 \rangle$$

In terms of these Vectors, as one is in the following figure is possible to be expressed any vector of the Plane of the following form: $v = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = v_1 \langle 1,0 \rangle + v_2 \langle 0,1 \rangle = v_1 i + v_2 j$



In where the Vector $v = v_1 i + v_2 j$ a linear combination is called of i and of j and you will scale them v_1 and v_2 they are called respectively, Horizontal Component and Component Vertical of v.

Canonical Unitary vectors in the Space

In the Space the Vectors are denoted as it were said previously by ordered trios $v = \langle v_1, v_2, v_3 \rangle$. Vector zero is denoted by $0 = \langle 0,0,0 \rangle$. Using the Unitary Vectors $i = \langle 1,0,0 \rangle$ $j = \langle 0,1,0 \rangle$ and $k = \langle 0,0,1 \rangle$ in the direction of z-axis, therefore the canonical annotation in terms of Unitary Vectors for a Vector V is:

$$\|pq\| = v = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$$

Naturally operations with the Vectors can be made in the Space.

Extreme of Vectors in the Space (it gives another Vector us)

Yes To it is equal a $\overline{A} = A_1\overline{i} + A_2\overline{j} + A_3\overline{k}$ and $\overline{B} = B_1\overline{i} + B_2\overline{j} + B_3\overline{k}$ then

$$\overline{A} + \overline{B} = (A_1 + B_1)\overline{i} + (A_2 + B_2)\overline{j} + (A_3 + B_3)\overline{k}$$

Subtraction of Vectors in the Space (it gives another Vector us)

Yes To it is equal a $\overline{A} = A_1\overline{i} + A_2\overline{j} + A_3\overline{k}$ and $\overline{B} = B_1\overline{i} + B_2\overline{j} + B_3\overline{k}$ then

$$\overline{A}-\overline{B}=(A_1-B_1)\overline{i}+(A_2-B_2)\overline{j}+(A_3-B_3)\overline{k}$$

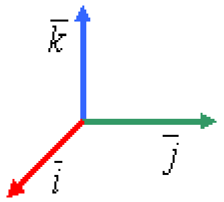
Multiplication of a vector by a scalar

(it gives another Vector us) and the Distributive property is used

If To it is equal a $\overline{A}=A_1\overline{i}+A_2\overline{j}+A_3\overline{k}$ and we are multiplied by scaling p would have left, another Vector

$$p\overline{A}=pA_1\overline{i}+pA_2\overline{j}+pA_3\overline{k}$$

Product To climb between Unitary Vectors



$$\overline{i}\bullet\overline{i}=\left\|\overline{i}\right\|^2=1$$

$$\overline{i}\bullet\overline{j}=\left\|\overline{i}\right\|\left|\overline{j}\right|\cos 90^\circ=0$$

$$\overline{i}\bullet\overline{k}=\left\|\overline{i}\right\|\left\|\overline{k}\right\|\cos 90^\circ=0$$

$$\overline{j}\bullet\overline{i}=\left\|\overline{j}\right\|\left|\overline{i}\right|\cos 90^\circ=0$$

$$\overline{j}\bullet\overline{j}=\left\|\overline{j}\right\|^2=1$$

$$\overline{j}\bullet\overline{k}=\left|\overline{j}\right|\left|\overline{k}\right|\cos 90^\circ=0$$

$$\overline{k}\bullet\overline{i}=\left\|\overline{k}\right\|\left\|\overline{i}\right\|\cos 90^\circ=0$$

$$\overline{k}\bullet\overline{j}=\left|\overline{k}\right|\left|\overline{j}\right|\cos 90^\circ=0$$

$$\overline{k}\bullet\overline{k}=\left\|\overline{k}\right\|^2=1$$

Examples:

- $2\bar{i} \bullet 5\bar{j} = 2 \bullet 5(\bar{i} \bullet \bar{j}) = 0$
- $(-3\bar{j}) \bullet 4\bar{j} = (-3) \bullet 4(\bar{j} \bullet \bar{j}) = -12 \bullet 1 = -12$

Definition of the Product Point (To climb or Internal)

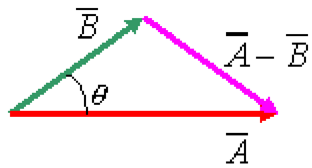
Up to here one has studied three operations with Vectors, the Sum and Subtraction of two Vectors and the Multiplication of a Vector by Scalar, which give by result a Vector. From one third operation will be introduced here the Product To climb (Internal Point or) whose result is not a vector but Scalar (a Number).

Note: The Product Point we are going it to denote by a Point $\bullet$

If we have two Vectors $\overline{A} = a_1 i + a_2 j + a_3 k$ and $\overline{B} = b_1 i + b_2 j + b_3 k$ that they form an angle θ to each other, it is possible to be said then by definition:

$$\overline{A} \bullet \overline{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and if we used the law of the Cosines



$$\left(\left\|\overline{A} - \overline{B}\right\|\right)^2 = \left(\left\|\overline{A}\right\|\right)^2 + \left(\left\|\overline{B}\right\|\right)^2 - 2\left\|\overline{A}\right\|\left\|\overline{B}\right\|\cos\theta$$

if we elevated to the Square the left side we would have left

$$\left(\left\|\overline{A}\right\|\right)^2 - 2\overline{A} \bullet \overline{B} + \left(\left\|\overline{B}\right\|\right)^2 = \left(\left|\mathbf{A}\right|\right)^2 + \left(\left|\mathbf{B}\right|\right)^2 - 2\left\|\overline{A}\right\|\left\|\overline{B}\right\|\cos\theta$$

cancelling similar terms

$$\overline{A} \bullet \overline{B} = \left\|\overline{A}\right\|\left\|\overline{B}\right\|\cos\theta \text{ in where } 0^\circ \leq \theta \leq 180^\circ$$

and the Angle θ it is obtained from the following way

$$\cos\theta = \frac{\overline{A} \bullet \overline{B}}{\left\|\overline{A}\right\|\left\|\overline{B}\right\|}$$

With the following example the doubts will be clarified that could have arisen

To find the Angle that exists both between following Vectors:

$$\overline{A} = 3i + 4j + 5k$$
$$\overline{B} = 2i - 3j + 4k$$

Solution: as Formula were said previously to find the Angle is:

$$Cos\,\theta = \frac{\overline{A} \bullet \overline{B}}{\left\|\overline{A}\right\|\left\|\overline{B}\right\|}$$

firstly we will calculate the Product Point $A \bullet B$

$$\overline{A} \bullet \overline{B} = (3)(2) + (4)(-3) + (5)(4) = 6 - 12 + 20 = 14$$

now we will calculate the Product of the Magnitudes

$$\left\|\overline{A}\right\|\left\|\overline{B}\right\| = \left(\sqrt{(3)^2 + (4^2) + (5)^2}\right)\left(\sqrt{(2)^2 + (-3)^2 + (4)^2}\right) =$$
$$= \left(\sqrt{50}\right)\left(\sqrt{29}\right) = \sqrt{(50)(29)} =$$
$$\therefore$$
$$\left\|\overline{A}\right\|\left\|\overline{B}\right\| = \sqrt{1450}$$

replacing these Values in Formula of the Cosine and clearing θ

$$Cos\theta = \frac{\overline{A} \bullet \overline{B}}{\|\overline{A}\|\|\overline{B}\|} = \frac{14}{\sqrt{1450}} = \frac{14}{38.07} \approx 0.367658$$

$\therefore$

$$\theta = Cos^{-1} 0.367658$$

$$\theta = 68.42^{\circ}$$

$ó$

$$\theta = 68^{\circ} 25' 43.48''$$

Properties of the Product Point (To climb or Internal)

If we have two Vectors $\overline{A} = a_1 i + a_2 j + a_3 k$ and $\overline{B} = b_1 i + b_2 j + b_3 k$ and if we carried out the Multiplication $\overline{A} \bullet \overline{B}$ we have left:

$$\overline{A} \bullet \overline{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

on the other hand if We multiplied $\overline{B} \bullet \overline{A}$ we have left:

$$\overline{B} \bullet \overline{A} = b_1 a_1 + b_2 a_2 + b_3 a_3$$

what it indicates to us that $\overline{A} \bullet \overline{B} = \overline{B} \bullet \overline{A}$ it is **the Commutative Property**

On the other hand if we have Scalar p and Multiplicands by $\left(\overline{A} \bullet \overline{B}\right)$ we would have left:

$$p\left(\overline{A} \bullet \overline{B}\right) = p\overline{A} \bullet \overline{B} = \overline{A} \bullet p\overline{B}$$ that it is **the Associative Property**

If we have two numbers You will scale p and q and Multiplicands by $\left(\overline{A} \bullet \overline{B}\right)$ we would have left:

$$pq\left(\overline{A} \bullet \overline{B}\right) = pq\overline{A} \bullet \overline{B} = \overline{A} \bullet pq\overline{B} = p\overline{A} \bullet q\overline{B} = q\overline{A} \bullet p\overline{B} = pq\left(\overline{A} \bullet \overline{B}\right)$$

Associative Property

Yes $\theta = 0^{\circ}$ $\overline{A} \bullet \overline{B} = \|\overline{A}\|\|\overline{B}\|$

$\theta = 90^{\circ} \perp \overline{A} \bullet \overline{B} = 0$ in this case the Vectors are Orthogonal

$\theta = 180^{\circ}$ $\qquad \overline{A} \bullet \overline{B} = -\|\overline{A}\|\|\overline{B}\|$

If We multiplied a Vector by itself, the Angle that forms among them is of 0° therefore:

$$\overline{A} \bullet \overline{A} = \|\overline{A}\|\|\overline{A}\|\cos 0^{\circ} \text{ and like him } \cos 0^{\circ} = 1$$

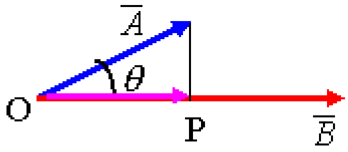
$$\overline{A}^{2} = \overline{A} \bullet \overline{A} = \|\overline{A}\|\|\overline{A}\| = \|\overline{A}\|^{2}$$

.

Vectors 4

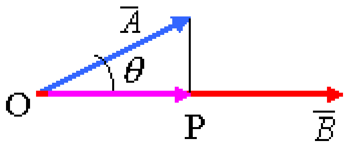
Components

The component of a Vector $\overline{A}$ in the direction of the Vector $\overline{B}$ it is Scalar that is obtained from the following way



In where the component of the Vector $\overline{A}$ on the Vector $\overline{B}$ he is equal to the Vector $\overline{OP}$ it is $Comp.\overline{A}/_{\overline{B}} = \overline{OP}$

Analytically the Component of a Vector $\overline{A}$ in the direction of the Vector $\overline{B}$ it is obtained from the following way:



like him $\cos\theta = \dfrac{\overline{OP}}{\left\|\overline{A}\right\|}$

clearing the Vector $\overline{OP}$ we would have left:

$$\overline{OP} = \left|\overline{A}\right|\cos\theta$$

and like $Comp.\overline{A}/_{\overline{B}} = \overline{OP}$ therefore.

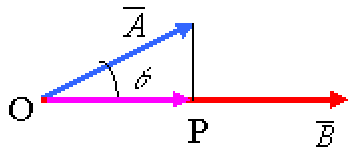
$$Comp.\overline{A}/_{\overline{B}} = \left\|\overline{A}\right\|\cos\theta$$

Of equal way we will be able to find the component of a Vector $\overline{B}$ on a Vector $\overline{A}$

$$Comp.\overline{B}/_{\overline{A}} = \left\|\overline{B}\right\| \cos\theta$$

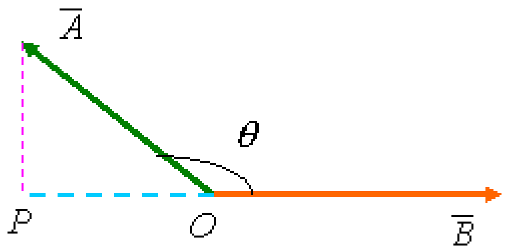
If the Component is Positive the Angle formed by the Vectors he is Acute and the Component has the same sense that the Vector $\overline{B}$.

$0^{\circ} \leq \theta \leq 90^{\circ}$

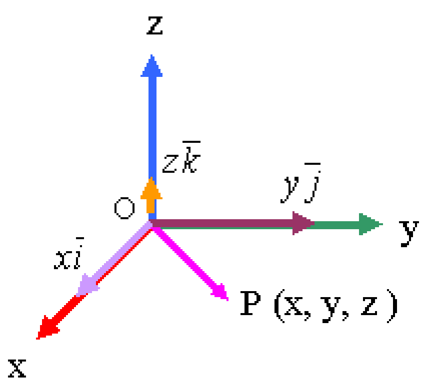


If the Component is Negative the Angulo formed by the Vectors is Obtuse and the Component has sense in opposition to the Vector $\overline{B}$.

$90^{\circ} \leq \theta \leq 180^{\circ}$



Projections



If we have a Vector $\overline{OP}$ in the Space and we want to see its Projections

In where the Projections (component) Vectorial of the Vector $\overline{OP}$ they are:

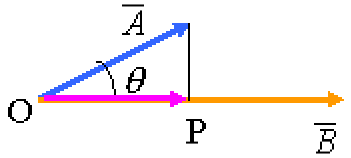
$x\bar{i}$ the projection of the Vector $\overline{OP}$ on the Y-axis of the Xs,

$y\bar{j}$ the projection of the Vector $\overline{OP}$ on the Y-axis of and and

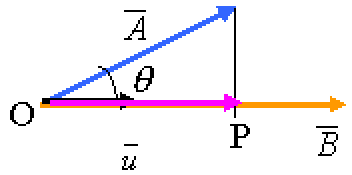
$z\bar{k}$ the Projection of the Vector $\overline{OP}$ on the Y-axis of the Z.

Vector of Position is called to that Vector that has the Point of Beginning in the Origin (Or) and in where Full stop of this Vector it indicates the position to us of a P point .

The Projection of the Vector $\overline{A}$ ($\overline{OP}$ in the direction of the Vector $\overline{B}$ it is Scalar that is obtained projecting the Vector $\overline{A}$ on the Vector $\overline{B}$



Analytically the Projection of the Vector $\overline{A}$ ($\overline{OP}$) on the Vector $\overline{B}$ it is obtained from the following way:



$$\operatorname{Proy.}\overline{A}/\overline{B} = \left(comp.\overline{A}/\overline{B}\right)\overline{u} = \overline{OP}\overline{u}$$

in where

And like $\bar{u} = \dfrac{\overline{B}}{\left\|\overline{B}\right\|}$ and $Comp.\overline{A}/\overline{B} = \overline{OP} = \left\|\overline{A}\right\|\cos\theta$

$$\operatorname{Proy.}\overline{A}/\overline{B} = \left(\left\|\overline{A}\right\|\cos\theta\right)\overline{u}$$

in the same way the Projection of a Vector can be found $\overline{B}$ on a Vector $\overline{A}$

$$\operatorname{Proy.}\overline{B}/\overline{A} = \left(\left\|\overline{B}\right\|\cos\theta\right)\overline{u}$$

NOTE: It is very common to confuse a Component that is Scalar, with a Projection that is a Vector.

Applications of the Product Point (To climb or Internal)

a) To find the Projection of the Vector $\overline{A} = 3\overline{i} - 2\overline{j} - 2\overline{k}$ on the Vector $\overline{B} = 4\overline{i} + 4\overline{j} - 7\overline{k}$

$$\mathrm{Pr}oy.\overline{A}/_{\overline{B}} = \overline{OPu} = \left(\|\overline{A}\| Cos\,\theta\right)\overline{u}$$

remembren that $\cos\theta = \dfrac{\overline{A} \bullet \overline{B}}{\|\overline{A}\|\|\overline{B}\|}$ and that $\overline{u} = \dfrac{\overline{B}}{\|\overline{B}\|}$

$$\mathrm{Pr}oy.\overline{A}/_{\overline{B}} = \overline{OPu} = \left(\|\overline{A}\| Cos\,\theta\right)\overline{u} = \left(\|\overline{A}\| \frac{\overline{A} \bullet \overline{B}}{\|\overline{A}\|\|\overline{B}\|}\right)\overline{u} = \left(\overline{A} \bullet \overline{u}\right)\overline{u}$$

$$\|\overline{B}\| = \sqrt{a^2 + b^2 + c^2} = \sqrt{4^2 + 4^2 + (-7)^2} = \sqrt{81} = 9$$

$$\overline{u} = \frac{\overline{B}}{\|\overline{B}\|} = \frac{4\overline{i} + 4\overline{j} - 7\overline{k}}{9} = \frac{4}{9}\overline{i} + \frac{4}{9}\overline{j} - \frac{7}{9}\overline{k}$$

$$\overline{A} \bullet \overline{u} = \left(3\overline{i} - 2\overline{j} - 2\overline{k}\right) \bullet \left(\frac{4}{9}\overline{i} + \frac{4}{9}\overline{j} - \frac{7}{9}\overline{k}\right) = \frac{12}{9} - \frac{8}{9} + \frac{14}{9} = \frac{18}{9} = 2$$

$$proy.\overline{A}/_{\overline{B}} = \left(\overline{A} \bullet \overline{u}\right)\overline{u} = 2\left(\frac{4}{9}\overline{i} + \frac{4}{9}\overline{j} - \frac{7}{9}\overline{k}\right)$$

$$proy.\overline{A}/_{\overline{B}} = \frac{8}{9}\overline{i} + \frac{8}{9}\overline{j} - \frac{14}{9}\overline{k}$$

b) To find the Component of the Vector $\overline{A} = 3\overline{i} - 2\overline{j} - 2\overline{k}$ on

Vector $\overline{B} = 4\overline{i} + 4\overline{j} - 7\overline{k}$

$$Comp.\overline{A}/_{\overline{B}} = \|\overline{A}\| \cos\theta$$

remembren that $Cos\theta = \dfrac{\overline{A} \bullet \overline{B}}{\left\|\overline{A}\right\|\left\|\overline{B}\right\|}$

$$Comp.\overline{A}/_{\overline{B}} = \left(\left\|\overline{A}\right\| Cos\,\theta\right) = \left(\left\|\overline{A}\right\| \frac{\overline{A} \bullet \overline{B}}{\left\|\overline{A}\right\|\left\|\overline{B}\right\|}\right) = \frac{\overline{A} \bullet \overline{B}}{\left\|\overline{B}\right\|}$$

$$Comp.\overline{A}/_{\overline{B}} = \frac{\langle 3,-2,-2\rangle \bullet \langle 4,4,-7\rangle}{\sqrt{(4)^2 + (4)^2 + (-7)^2}} = \frac{12 - 8 + 14}{\sqrt{81}} = \frac{18}{9} = 2$$

.

vectors

Definition of the Product Cross (Vectorial)

Given two Vectors $\overline{A}$ and $\overline{B}$ Vectorial Product or Product is defined as Cross

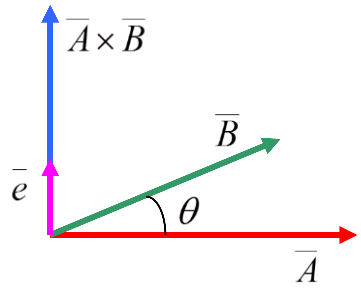
$$\overline{A}\times\overline{B}=\overline{e}\left(\left\|\overline{A}\right\|\left\|\overline{B}\right\| sen\,\theta\right) \qquad 0^\circ \le \theta \le 180^\circ$$

in where $\overline{e}$ it is a perpendicular Vector to Plane of the Vectors $\overline{A}$ and $\overline{B}$ and in where $\overline{e}$ it is a Unitary Vector with the same direction and sense of the Vector $\overline{A}\times\overline{B}$

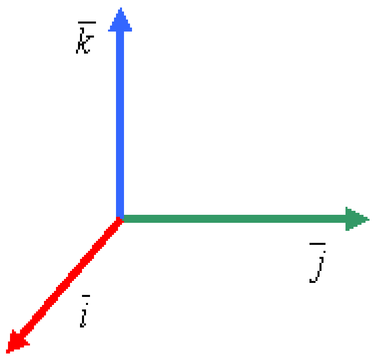
$\overline{A}\times\overline{B}$ it is $\perp$ a $\overline{A}$

$\overline{A}\times\overline{B}$ it is $\perp$ a $\overline{B}$

$$\overline{e}=\frac{\overline{A}\times\overline{B}}{\left\|\overline{A}\times\overline{B}\right\|}$$



Product of unit vectors vectorial $\overline{i},\overline{j},\overline{k}$



$\overline{i}\times\overline{i}=\overline{0}$

$\overline{i}\times\overline{j}=\overline{k}$

$\overline{i}\times\overline{k}=-\overline{j}$

$\overline{j}\times\overline{i}=-\overline{k}$

$\overline{j}\times\overline{j}=\overline{0}$

EXERCISES

I It locates the position of points in space:

1) $A(3, 2, 5)$
2) $B(-4, -3, 6)$
3) $C(2, -4, -3)$
4) $D(-3, 3, -4)$

II In the coordinate plane represent the vectors:

1) $\boldsymbol{A} = \langle 4, -3, 2 \rangle$
2) Initial Point $P(0, 3, 4)$ y final Point $Q(3, -4, 0)$.

III Calculate the magnitude of the vector that starts at the point P (4, 5, - 3) and ends at the point Q (- 2, - 1, 2).

IV Given vectors A = ⟨-2, 3, 5⟩ , B = ⟨ 1, - 5, 4⟩ and C = ⟨7, 0, 3⟩ get:

1) $\boldsymbol{A} + \boldsymbol{B}$
2) $\boldsymbol{B} - \boldsymbol{C}$
3) $-2\boldsymbol{B}$
4) $3\boldsymbol{C} - 2\boldsymbol{A}$
5) $\frac{1}{2}\boldsymbol{A} + \frac{1}{2}\boldsymbol{B} - \frac{3}{2}\boldsymbol{C}$
6) $\frac{1}{3}(\boldsymbol{A} - \boldsymbol{B}) + \frac{2}{3}\boldsymbol{C}$
7) $\| \boldsymbol{B} - \boldsymbol{C} \|$

V Be A = 3i-3j + 4 k and the vector B = 2i + 6j + 3 k get:

1) $\boldsymbol{A} \bullet \boldsymbol{B}$
2) $\boldsymbol{A} \ X \ \boldsymbol{B}$
3) **The angle between the vectors A and B.**

VI **Obtain the projection of the vector A = ⟨-2, 1, 2⟩ in vector B = ⟨ 4, - 4, 2⟩ .**

VII Si $\boldsymbol{A} = \langle 4, -3, 1 \rangle$ y $\boldsymbol{B} = \langle -2, 2, 1 \rangle$ obtain:

1) $\boldsymbol{A} \, X \, \boldsymbol{B}$
2) $\boldsymbol{U}_{\boldsymbol{A} \, X \, \boldsymbol{B}}$

VIII Obtain $\boldsymbol{A} \ \boldsymbol{X} \ \boldsymbol{B} \bullet \ \boldsymbol{C}$ si $\boldsymbol{A} = \langle 3, 4, -2 \rangle$, $\boldsymbol{B} = \langle -1, 2, 3 \rangle$ y $C = \langle -2, 2, 1 \rangle$.

SOLUTIONS

I

II

III $\|\boldsymbol{PQ}\| = 7$

IV

1) $\boldsymbol{A} + \boldsymbol{B} = \langle -1, -2, -1 \rangle$

2) $\boldsymbol{B} - \boldsymbol{C} = \langle -6, -5, 7 \rangle$

3) $-2\boldsymbol{B} = \langle -2, 10, -8 \rangle$

4) $3\boldsymbol{C} - 2\boldsymbol{A} = \langle 25, -6, 1 \rangle$

5) $\frac{1}{2}\boldsymbol{A} + \frac{1}{2}\boldsymbol{B} - \frac{3}{2}\boldsymbol{C} = \langle -11, -1, 4 \rangle$

6) $\frac{1}{3}(\boldsymbol{A} - \boldsymbol{B}) + \frac{2}{3}\boldsymbol{C} = \langle \frac{11}{3}, \frac{8}{3}, -5 \rangle$

7) $\|\boldsymbol{B} - \boldsymbol{C}\| = \sqrt{110}$

V

1) $\boldsymbol{A} \bullet \boldsymbol{B} = 0$

2) $\boldsymbol{A}\, X\, \boldsymbol{B} = \langle -33, -1, 24 \rangle$

3) $\theta = 90^0$

VI $\boldsymbol{Proy_B A} = \langle -\frac{\mathbf{8}}{\mathbf{9}}, \frac{\mathbf{8}}{\mathbf{9}}, -\frac{\mathbf{4}}{\mathbf{9}} \rangle$

VII

1) $\boldsymbol{A}\, X\, \boldsymbol{B} = \langle -5, -6, 2 \rangle$

2) $\boldsymbol{U}_{\boldsymbol{A}\, X\, \boldsymbol{B}} = \frac{\langle -5, -6, 2 \rangle}{\sqrt{65}}$

VIII $\boldsymbol{A}\ \boldsymbol{X}\ \boldsymbol{B} \bullet\ \boldsymbol{C} = \boldsymbol{-36}$

$$\overline{j} \times \overline{k} = \overline{i}$$

$$\overline{k} \times \overline{i} = \overline{j}$$

$$\overline{k} \times \overline{j} = -\overline{i}$$

$$\overline{k} \times \overline{k} = \overline{0}$$

When They are multiplied Unitary Vectors it is important to make notice the following thing:

· If the Product is in favor of the small hands of the clock he is NEGATIVE

· If the product is against the small hands of the clock he is POSITIVE

If we have two Vectors $\overline{A} = A_1\overline{i} + A_2\overline{j} + A_3\overline{k}$ and $\overline{B} = B_1\overline{i} + B_2\overline{j} + B_3\overline{k}$ and we carried out the product $\overline{A} \times \overline{B}$ we would have left:

$$\overline{A} \times \overline{B} = \overline{i}\left(a_2b_3 - b_2a_3\right) - \overline{j}\left(a_1b_3 - a_3b_1\right) + \overline{k}\left(a_1b_2 - a_2b_1\right)$$

this we can easily demonstrate it either by a simple multiplication or developing the method of MINORS of a determinant by the first Row

$$\overline{A} \times \overline{B} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \overline{i}\left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\right) - \overline{j}\left(\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\right) + \overline{k}\left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)$$

$$\overline{A} \times \overline{B} = \overline{i}\left(a_2b_3 - b_2a_3\right) - \overline{j}\left(a_1b_3 - a_3b_1\right) + \overline{k}\left(a_1b_2 - a_2b_1\right)$$

So that the product of two Vectors is not Zero these Vectors do not have to be Parallel, because if is Parallel the angle that forms among them or is 0° (so that they are in the same Direction) or it is of 180° (because they are in opposite sense) and if we remembered it formulates it of the product that is

$$\overline{A} \times \overline{B} = \overline{e}\left(\left\|\overline{A}\right\|\left\|\overline{B}\right\| sen\, \theta\right)$$

and like the sine of 0° and the sine of 180° they are equal to 0

$$\overline{A}\times\overline{B}=\bar{e}\left(\left|\overline{A}\right|\left|\overline{B}\right|\times 0\right)=0$$

Properties of the Product Cross (Vectorial)

- **The product of a Vector by itself is zero** $\overline{A}\times\overline{A}=0$
- **The product of a Vector by the null Vector is zero** $\overline{A}\times\phi=0$

$$\overline{A}\times\left(\overline{B}+\overline{C}\right)=\overline{A}\times\overline{B}+\overline{A}\times\overline{C}$$ **Law Distributive**

$$m\times\left(\overline{A}+\overline{B}\right)=m\overline{A}\times\overline{B}=\overline{A}\times m\overline{B}$$ **Law Distributive**

If the Vectors $\overline{A}$ and $\overline{B}$ this given is the contiguous sides of a then PARALLELOGRAM the area of this parallelogram by:

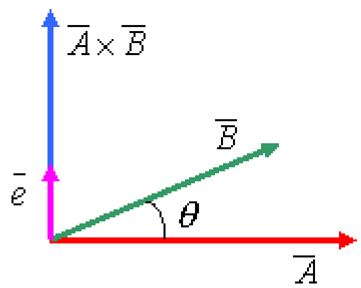
$$A_p=\left\|\overline{A}\times\overline{B}\right\|$$

If the Vectors $\overline{A}$ and $\overline{B}$ they are the contiguous sides of a then TRIANGLE the area of the triangle is equal a:

$$A_t=\frac{1}{2}\left\|\overline{A}\times\overline{B}\right\|$$

so that it is but clear we will explain it with an EXAMPLE

If the Points $P(5,4,5)$ $Q(4,6,10)$ and $R(1,8,7)$ they are the Vertices of a Triangle to find his Area.



Solution:

$$\overline{PQ} = (4-5)\bar{i} + (6-4)\bar{j} + (10-5)\bar{k} = -\bar{i} + 2\bar{j} + 5\bar{k}$$

$$\overline{PR} = (1-5)\bar{i} + (8-4)\bar{j} + (7-5)\bar{k} = -4\bar{i} + 4\bar{j} + 2\bar{k} \qquad \overline{PQ}\times\overline{PR} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 2 & 5 \\ -4 & 4 & 2 \end{vmatrix}$$

$$\overline{PQ}\times\overline{PR} = \bar{i}\left(\begin{vmatrix} 2 & 5 \\ 4 & 2 \end{vmatrix}\right) - \bar{j}\left(\begin{vmatrix} -1 & 5 \\ -4 & 2 \end{vmatrix}\right) + \bar{k}\left(\begin{vmatrix} -1 & 2 \\ -4 & 4 \end{vmatrix}\right)$$

$$\overline{PQ}\times\overline{PR} = -16\bar{i} - 18\bar{j} + 4\bar{k}$$

$$\left\|\overline{PQ}\times\overline{PR}\right\| = \sqrt{(-16)^2 + (-18)^2 + (4)^2} = \sqrt{256+324+16} = \sqrt{596}$$

$$A_t = \frac{1}{2}\left\|\overline{PQ}\times\overline{PR}\right\| = \frac{1}{2}\sqrt{596} = \frac{\sqrt{596}}{2} = aprox. = 12.2u^2$$

Triple Product To climb

Given the Vectors $\overline{A}$ $\overline{B}$ and $\overline{C}$ we have the Triple Product To climb is defined as:

$$\overline{A}\bullet\overline{B}\times\overline{C} = ESCALAR$$

$$\overline{A}\bullet\overline{B}\times\overline{C} = \overline{B}\bullet\overline{C}\times\overline{A} = \overline{C}\bullet\overline{A}\times\overline{B}$$

or

its NEGATIVE $\overline{A}\bullet\overline{C}\times\overline{B} = \overline{B}\bullet\overline{A}\times\overline{C} = \overline{C}\bullet\overline{B}\times\overline{A}$

If the Vectors $\overline{A}$ $\overline{B}$ and $\overline{C}$ this dice is the contiguous sides of a then PARALELEPÍPEDO the Volume of this Parallelepiped by:

$$V_p = \left|\overline{A}\bullet\overline{B}\times\overline{C}\right|$$

If the Vectors $\overline{A}$ $\overline{B}$ and $\overline{C}$ they are the contiguous sides of a TETRAHEDRON (pyramid of 4 triangular sides) then the Volume of this Tetrahedron this dice by:

$$V_p = \frac{1}{6}\left|\overline{A} \bullet \overline{B} \times \overline{C}\right|$$

If three Vectors are COPLANARIOS (in the same Plane) then their triple product TO CLIMB he is equal to ZERO.

$$\overline{A} \bullet \overline{B} \times \overline{C} = 0$$

If two of the three Vectors $\overline{A}$ $\overline{B}$ and $\overline{C}$ they are Parallel then the Triple Product To climb is equal to ZERO.

$\overline{A}$ it is parallel a $\overline{B}$

$\overline{A}$ it is parallel a $\overline{C}$

$\overline{B}$ it is parallel a $\overline{C}$

Example: To carry out the following Product $\overline{A} \bullet \overline{B} \times \overline{C}$ if $\overline{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$

$\overline{B} = B_1\bar{i} + B_2\bar{j} + B_3\bar{k}$ and $\overline{C} = C_1\bar{i} + C_2\bar{j} + C_3\bar{k}$ (first one takes place $\overline{B} \times \overline{C}$)

$$\overline{B} \times \overline{C} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \bar{i}\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - \bar{J}\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + \bar{K}\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

$$\overline{A} \bullet \overline{B} \times \overline{C} = \left(A_1\bar{i} + A_2\bar{j} + A_3\bar{k}\right) \bullet \bar{i}\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - \bar{j}\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + \bar{k}\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

And recalling that the multiplication of unit vectors

$$\bar{i} \bullet \bar{i} = \left\|\bar{i}\right\|^2 = 1$$

$$\bar{i} \bullet \bar{j} = \left\|\bar{i}\right\|\left\|\bar{j}\right\|\cos 90^\circ = 0$$

$$\bar{i} \bullet \bar{k} = \left\|\bar{i}\right\|\left\|\bar{k}\right\|\cos 90^\circ = 0$$

$$\bar{j} \bullet \bar{i} = \|\bar{j}\|\|\bar{i}\|\cos 90^\circ = 0$$

$$\bar{j} \bullet \bar{j} = \|\bar{j}\|^2 = 1$$

$$\bar{j} \bullet \bar{k} = \|\bar{j}\|\|\bar{k}\|\cos 90^\circ = 0$$

$$\bar{k} \bullet \bar{i} = \|\bar{k}\|\|\bar{i}\|\cos 90^\circ = 0$$

$$\bar{k} \bullet \bar{j} = \|\bar{k}\|\|\bar{j}\|\cos 90^\circ = 0$$

$$\bar{k} \bullet \bar{k} = \|\bar{k}\|^2 = 1$$

we would have left:

$$\bar{A} \bullet \bar{B} \times \bar{C} = A_1 \begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - A_2 \begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + A_3 \begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

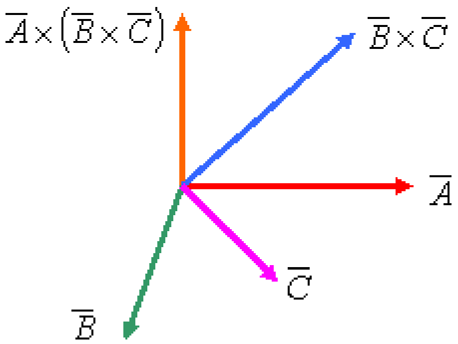
which can be written of the following way

$$\bar{A} \bullet \bar{B} \times \bar{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Triple Vectorial Product

Given the Vectors $\bar{A}$ $\bar{B}$ and $\bar{C}$ we have the Triple Vectorial Product is defined as:

$$\bar{A} \times (\mathbf{B} \times \mathbf{C}) = Vector$$



In where the Vector $\overline{A}\times(\mathbf{B}\times\mathbf{C})$ it is coplanar and Perpendicular to the Vector $\overline{A}$ and to the Vector $(\mathbf{B}\times\mathbf{C})$

The Vector $\overline{B}\times\overline{C}$ he is Perpendicular to the Vectors $\overline{A}$ and $\overline{B}$

$$\overline{A}\times(\overline{B}\times\overline{C})\neq(\overline{A}\times\overline{B})\times\overline{C}$$ (it does not obey the law Commutative)

Rule of the Average Factor

$$\overline{A}\times(\overline{B}\times\overline{C})=\overline{B}(\overline{A}\bullet\overline{C})-\overline{C}(\overline{A}\bullet\overline{B})$$

$$(\overline{A}\times\overline{B})\times\overline{C}=\overline{B}(\overline{A}\bullet\overline{C})-\overline{A}(\overline{B}\bullet\overline{C})$$

With the following example we will clarify the doubts.

If we have the Vectors $\overline{A}=3\overline{i}+5\overline{j}+7\overline{k}$ $\overline{B}=-\overline{i}+2\overline{j}-3\overline{k}$ and $\overline{C}=3\overline{i}+4\overline{j}+5\overline{k}$ to find $\overline{A}\times(\overline{B}\times\overline{C})$

$$\overline{A}\times(\overline{B}\times\overline{C})=\overline{B}(\overline{A}\bullet\overline{C})-\overline{C}(\overline{A}\bullet\overline{B})$$

$$\overline{A}\bullet\overline{C}=(3\overline{i}\times3\overline{i})+(5\overline{j}\times4\overline{j})+(-7\overline{k}\times5\overline{k})=9+20-35=-6$$

$$\overline{A}\bullet\overline{B}=(3\overline{i}\times-\overline{i})+(5\overline{j}\times2\overline{j})+(-7\overline{k}\times-3\overline{k})=-3+10+21=28$$

$$\overline{A}\times(\overline{B}\times\overline{C})=(-\overline{i}+2\overline{j}-3\overline{k})(-6)-(3\overline{i}+4\overline{j}+5\overline{k})(28)=$$

$$\therefore$$

$$\overline{A}\times(\overline{B}\times\overline{C})=6\overline{i}-12\overline{j}+18\overline{k}-84\overline{i}-112\overline{j}-140\overline{k}$$

$$\overline{A}\times(\overline{B}\times\overline{C})=-78\overline{i}-124\overline{j}-122\overline{k}$$

Applications of the Product Cross

To find the Volume of the Parallelepiped that has like contiguous edges the following Vectors,

$$\overline{A} = -\bar{i} + 2\overline{j} + 5\overline{k} \;,\; \overline{B} = -4\bar{i} + 4\overline{j} + 2\overline{k} \text{ and } \overline{C} = -3\bar{i} + 2\overline{j} + 4\overline{k}$$

$$V_P = \overline{A} \bullet \overline{B} \times \overline{C} = \begin{vmatrix} -1 & 2 & 5 \\ -4 & 4 & 2 \\ -3 & 2 & 4 \end{vmatrix} = -1\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} -4 & 2 \\ -3 & 4 \end{vmatrix} + 5\begin{vmatrix} -4 & 4 \\ -3 & 2 \end{vmatrix}$$

$$V_P = -1(16-4) - 2(-16+6) + 5(-8+12) = -12 + 20 + 20$$

$$V_P = 28$$

To find a Vector that has a magnitude of 25 and which he is Perpendicular to the Plane formed by the vectors, $\overline{A} = 4\bar{i} + 5\overline{j} - 6\overline{k}$ and $\overline{B} = 2\bar{i} + 4\overline{j} + 6\overline{k}$

Solution: carrying out the product Cross is its perpendicularity

$$\overline{A} \times \overline{B} = \begin{vmatrix} \bar{i} & \overline{j} & \overline{k} \\ 4 & 5 & -6 \\ 2 & 4 & 6 \end{vmatrix} = \bar{i}\begin{vmatrix} 5 & -6 \\ 4 & 6 \end{vmatrix} - \overline{j}\begin{vmatrix} 4 & -6 \\ 2 & 6 \end{vmatrix} + \overline{k}\begin{vmatrix} 4 & 5 \\ 2 & 4 \end{vmatrix} =$$

$$\therefore$$

$$\overline{A} \times \overline{B} = 54\bar{i} - 36\overline{j} + 6\overline{k}$$

calculating the Unitary Vector ($\overline{u}$ the direction and sense of the Vector) in

$$\overline{u} = \frac{\overline{A} \times \overline{B}}{\left\|\overline{A} \times \overline{B}\right\|} = \frac{54\bar{i} - 36\overline{j} + 6\overline{k}}{\sqrt{(54)^2 + (-36)^2 + (6)^2}} = \frac{54\bar{i} - 36\overline{j} + 6\overline{k}}{\sqrt{4248}}$$

$$\therefore$$

$$\overline{u}_{\overline{A} \times \overline{B}} = \frac{54}{\sqrt{4248}}\bar{i} - \frac{36}{\sqrt{4248}}\overline{j} + \frac{6}{\sqrt{4248}}\overline{k}$$

multiplying the Unitary Vector by 25 we have left:

$$25\bar{u} = 25\left(\frac{54}{\sqrt{4248}}\bar{i} - \frac{36}{\sqrt{4248}}\bar{j} + \frac{6}{\sqrt{4248}}\bar{k}\right) =$$

$$\therefore$$

$$25\bar{u} = \frac{1350}{\sqrt{4248}}\bar{i} - \frac{900}{\sqrt{4248}}\bar{j} + \frac{150}{\sqrt{4248}}\bar{k}$$

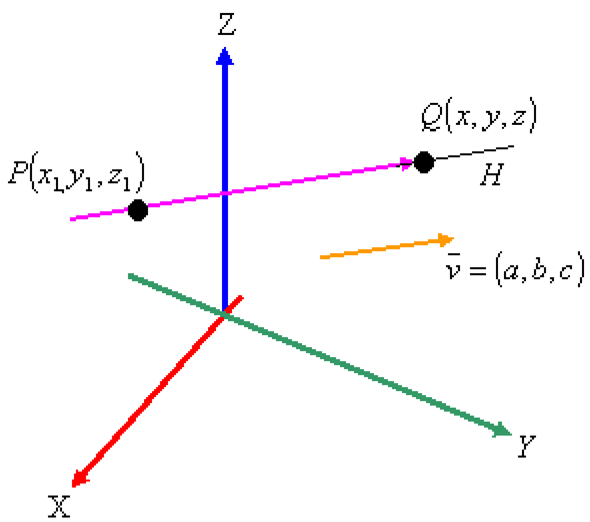
Note: A problem of this type has two solutions in agreement with the sense of the Vector (the other serious answer the same Vector but multiplied by -1)

The React and the Plane

Equation of the Straight line in Vectorial, Parametric and Symmetrical Forma

It forms Vectorial. Normally the Slope is used to express the Equation of a Straight line in a Plane, in the Space is more advisable to use **VECTORS**.

If we considered the straight H that happens through the point $P(x_1, y_1, z_1)$ and by the point $Q(x, y, z)$ and she is Parallel to the Vector $\bar{v} = \langle a, b, c \rangle$ in where the Vector $\bar{v}$ it is the Vector of Direction or Vector Director of the straight H and to, b, and cs are their numbers directors, that means that the Vector $\overline{PQ}$ it is a multiple To climb of $\bar{v}$ so that $\overline{PQ} = t\bar{v}$ where t it is Scalar (a Number)



In where **the Equation of a Straight line in the Space** this given by:

$$\overline{PQ} = \langle x - x_1, y - y_1, z - z_1 \rangle = t\bar{v} = t\langle a, b, c \rangle = \langle at, bt, ct \rangle$$

Parametric equations of a Straight line in the Space. In agreement with the previous equation if we equaled:

$$at = x - x_1$$

$$bt = y - y_1$$

$$ct = z - z_1$$

and now we cleared x, and, and z we obtain:

$$x = at + x_1$$

$$y = bt + y_1$$

$$z = ct + z_1$$

That they are **the Parametric Equations** (where t it is the parameter)

Symmetrical Equations. If of the three previous Parametric Equations we cleared t and we equaled them has left:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad a, b, c \neq 0$$

Distance of a Point in the Space to a Straight line

The distance of a Point Q to a Straight line in the Space is determined by:

$$D = \frac{\left\| \overline{PQ} \times \overline{PR} \right\|}{\left\| \overline{PR} \right\|}$$

in where $\overline{PR}$ it is a Vector of direction of Recta and P is a Point of the Straight line.

Example 1. Finds the distance from a Point Q (2,1,2) to the Straight line that happens through Points P (3,2,3) and R (2, -2, 1)

Solution:

Formula of the Distance is: $D = \frac{\left\| \overline{PQ} \times \overline{PR} \right\|}{\left\| \overline{PR} \right\|}$

Firstly we needed the Vectors Position and Direction which serein the Vectors formed by the points P and Q $\left(\overline{PQ}\right)$ Position and the points P and R $\left(\overline{PR}\right)$ Address

$$\overline{PQ} = \langle (2-3),(1-2),(2-3) \rangle = \langle -1,-1,-1 \rangle$$
$$\overline{PR} = \langle (2-3),(-2-2),(1-3) \rangle = \langle -1,-4,-2 \rangle$$

Carrying out the Product Cross

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ -1 & -4 & -2 \end{vmatrix} = i\begin{vmatrix} -1 & -1 \\ -4 & -2 \end{vmatrix} - j\begin{vmatrix} -1 & -1 \\ -1 & -2 \end{vmatrix} + k\begin{vmatrix} -1 & -1 \\ -1 & -4 \end{vmatrix}$$

$\therefore$

$$\overline{PQ} \times \overline{PR} = (2-4)i - (2-1)j + (4-1)k$$

$$\overline{PQ} \times \overline{PR} = \langle -2,-1,3 \rangle$$

$$D = \frac{\left\| \overline{PQ} \times \overline{PR} \right\|}{\left\| \overline{PR} \right\|} = \frac{\sqrt{(-2)^2 + (-1)^2 + (3)^2}}{\sqrt{(-1)^2 + (-4)^2 + (3)^2}} = \frac{\sqrt{14}}{\sqrt{26}} \approx 0.734$$

Example 2. To find the Distance of the Point Q(1, 2, 1) to the Straight line that this given by the following Parametric Equations

$x = 3 - 2t$, $y = 3t$ and $z = 4 + t$

Solution:

Formula of the Distance is: $D = \dfrac{\left\| \overline{PQ} \times \overline{PR} \right\|}{\left\| \overline{PR} \right\|}$

With the Parametric Equations we are going to both deduce points of the Straight line that we needed so that along with the Point (Q) said to be able to find the Distance

Remembering that the Equation of a Straight line in the Space this given by:

$$Ec.R = \langle x - x_1, y - y_1, z - z_1 \rangle = t\overline{v} = t\langle a,b,c \rangle = \langle at,bt,ct \rangle$$

$$x = 3 - 2t \text{ , } y = 3t \text{ and } z = 4 + t$$

And since the Vector $\bar{v} = \langle a,b,c \rangle$ we have then that $\bar{v} = \langle -2,3,1 \rangle$ and if we determined a Point of the Straight line with the Parametric Equations and knowing that t the Independent variable is the parameter () and choosing the simple value but it stops t that he is the 0 (zero) therefore the Point of the P Straight line we would have left that

$$x = 3 \quad y = 0 \quad z = 4 \quad P(3,0,4)$$

now if we calculated the Magnitude $\overline{PQ}$ we have left:

$$\overline{PQ} = \langle (x_q - x_p), (y_q - y_p), (z_q - z_p) \rangle = \langle (1-3), (2-0), (1-4) \rangle = \langle -2$$

calculating the product now Cross

$$\overline{PQ} \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & 2 & -3 \\ -2 & 3 & 1 \end{vmatrix} = 11\bar{i} + 8\bar{j} - 2\bar{k} = \langle 11,8,-2 \rangle$$

finally if we used it formulates it of the Distance

$$D = \frac{\left\| \overline{PQ} \times \bar{v} \right\|}{\left\| \bar{v} \right\|} = \frac{\sqrt{(11)^2 + (8)^2 + (-2)^2}}{\sqrt{(-2)^2 + (3)^2 + (1)^2}} = \frac{\sqrt{189}}{\sqrt{14}} \approx 3.67$$

The <u>react and the Plane</u>

To deduce the Equation of a Plane in Canonical and General Forma

The Plane that contains the Point (x_1, y_1, z_1) and it has a Normal Vector $\bar{n} = \langle a, b, c \rangle$ it can imagine in canonical Form by the following equation:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$ It forms Canonical

and if we regrouped the terms can be represented in General Form by the following equation:

$$ax + by + cz + d = 0$$ It forms General

in where to, b, c, and d is constant and the Perpendicular Vector (Normal) to the Plane this dice by:

$$\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$$

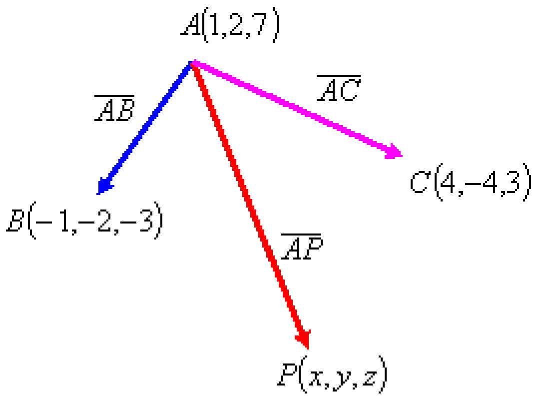
Example 1: the Normal Vector (Perpendicular) to the Plane $2x + 3y - 2z - 24 = 0$ it is:

$$\bar{n} = 2\bar{i} + 3\bar{j} - 2\bar{k}$$

Example 2: to find the Equation of the Plane that happens through the Points $A(1,2,7)$ $B(-1,-2,-3)$ and $C(4,-4,3)$

Solution:

If we had aside from those 3 points another point but, we could then form 3 Vectors that serein Coplanarios and if we remembered that the Triple Product To climb of three Coplanarios Vectors is equal to zero that that way it we will be able to find



It is understood that the P Point is an arbitrary Point that can be anywhere of the Plane therefore:

$$\overline{AP}\bullet\overline{AB}\times\overline{AC}=0$$

$$\overline{AP}=(x-1)\bar{i}+(y-2)\bar{j}+(z-7)\bar{k}$$

$$\overline{AB}=(-1-1)\bar{i}+(-2-2)\bar{j}+(-3-7)\bar{k}=-2\bar{i}-4\bar{j}-10\bar{k}$$

$$\overline{AC}=(4-1)\bar{i}+(-4-2)\bar{j}+(3-7)\bar{k}=3\bar{i}-6\bar{j}-4\bar{k}$$

$$\overline{AP}\bullet\overline{AB}\times\overline{AC}=\begin{vmatrix}(x-1) & (y-2) & (z-7)\\ -2 & -4 & -10\\ 3 & -6 & -4\end{vmatrix}$$

$$\overline{AP}\bullet\overline{AB}\times\overline{AC}=(x-1)\begin{vmatrix}-4 & -10\\ -6 & -4\end{vmatrix}-(y-2)\begin{vmatrix}-2 & -10\\ 3 & -4\end{vmatrix}+(z-7)\begin{vmatrix}-2 & -4\\ 3 & -6\end{vmatrix}$$

$$\overline{AP}\bullet\overline{AB}\times\overline{AC}=(x-1)(16-60)-(y-2)(8+30)+(z-7)(12+12)$$

$$\overline{AP}\bullet\overline{AB}\times\overline{AC}=(x-1)(-44)-(y-2)(38)+(z-7)(24)$$

$$\overline{AP}\bullet\overline{AB}\times\overline{AC}=-44x+44-38y+76+24z-168$$

$$\overline{AP}\bullet\overline{AB}\times\overline{AC}=-44x-38y+24z-48$$

Equation of the Plane That contains the Points To, B and C

Another way to solve the serious previous example:

Example 3: To find the Equation of the Plane that happens through the Points $A(1,2,7)$ $B(-1,-2,-3)$ and $C(4,-4,3)$

compute the range $\overline{AB}$ and the distance $\overline{AC}$

$$\overline{AB} = \langle -2,-4-10\rangle \text{ and } \overline{AC} = \langle 3,-6,-4\rangle$$

and if we found the Vector Normal

$$\bar{n} = \overline{AB} \times \overline{AC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & -4 & -10 \\ 3 & -6 & -4 \end{vmatrix} = -44\bar{i} - 38\bar{j} + 24\bar{k}$$

of where we concluded that $a = -44$ $b = -38$ and $c = 24$ and replacing these values in the Equation of the Plane that this in Canonical Forma

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$-44(x-1) - 38(y-2) + 24(z-7) = 0$$

$$-44x + 44 - 38y + 76 + 24z - 168 = 0$$

$$-44x - 38y + 24z - 48 = 0$$

Equation of the Plane That contains the Points To, B and C

Distance of a Point in the Space to a Plane

The distance of a Point $P_1(x_1, y_1, z_1)$ (no pertaining to the Plane) to the Plane $ax + by + cz + d = 0$ this given by the following relation:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 1: Compute the range of the Point $P(2,2,1)$ to the Plane $2x-3y+2z-5=0$

$$D=\frac{|2(2)+(-3)(2)+2(1)-5|}{\sqrt{(2)^2+(-3)^2+(2)^2}}=\frac{|4-6+2-5|}{\sqrt{4+9+4}}=\frac{|-5|}{\sqrt{17}}\approx 1.21$$

Example 2: Compute the range between two Parallel Planes

$$2x-3y+3z-8=0 \qquad y \qquad 2x-3y+3z+24=0$$

Solution: In this occasion a Point does not occur us, therefore we are going it to choose of the equation $2x-3y+3z-8=0$ assigning values a Y and to Z and clearing X we will find a P Point

if $y=0$ and $z=0$ replacing them in the chosen equation and clearing x we have left:

$$2x-3(0)+3(0)-8=0$$

$$x=\frac{8}{2}=4$$

therefore the found point is $P(4,0,0)$ and coming like in the previous example

$$D=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$$D=\frac{|2(4)+(-3)(0)+3(0)+24|}{\sqrt{(2)^2+(-3)^2+(3)^2}}=\frac{|8-0+0+24|}{\sqrt{4+9+9}}=\frac{|32|}{\sqrt{22}}\approx 6.822$$

EXERCISES

1) Obtain the parametric equations of the straight line that passes through (5, 1, - 3) and is parallel to the vector 〈8, 5, -2〉.

2) Obtain the distance of the point (- 1, 3 - 2) to the straight line of parametric equations x = 5, y = 3t, z = 3-2t.

3) Obtain the equation of the plane that passes through the point (3, - 2, 6) and is perpendicular to the vector 〈-2, 5, 1〉.

4) Obtain the equation of the plane passing through the points (2, - 6, - 1), (5, 0, 2), (4, 2, 5).

5) Obtain the distance of the point (2, 1, 3) the plane 2 x-4y + z = 1.

ANSWERS

1) $x = 5 + 8t$, $y = 1 + 5t$, $z = -3 - 2t$.

2) $d = \frac{3\sqrt{793}}{13}$

3) $2x - 5y - z = 10$

4) $12x - 12y + 12z = 84$

5) $d = \frac{2\sqrt{21}}{21}$